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On the K-theory of the loop space of a Lie group

by

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## Abstract

The thesis studies the K-theory of the loop space of compact Lie groups. There are three independent chapters.

The first chapter is devoted to proving the non-convergence of a K-theory Eilenberg-MacLane spectral sequence for the path space fibration on a Lie group with torsion in its cohomology.

The second chapter generalises a method of Bott. We obtain an algebraic expression for  $K^*(\Omega G)$  in terms of the representation theory of the group  $G$ . Explicit results are obtained for the unitary groups and the first symplectic group.

The third chapter develops and extends a method due to T. Petrie to find the Hopf algebra structure of the complex bordism of  $\Omega G$ . The Conner-Floyd isomorphism is used to obtain explicit results for  $K_*(\Omega G)$  for a larger class of groups than in chapter two.

### Erratum

There is a mistake in the proof of theorem (1.4.1). The second sentence on page 28 is false. Contrary to the stated result  $H$  need not be a divided polynomial algebra, and indeed fails to be in the corresponding ordinary cohomology spectral sequence. It is quite possible that, modulo decomposable elements,  $y_1$  may be divisible by an integer  $k \geq 2$ . Hence the argument of the rest of the proof fails.

Thus theorem (1.4.1) and its corollaries, (1.4.2) and (1.4.3), are not proved. The results of chapters 2 and 3 are independent, and so are unaffected.

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## Introduction

Let  $G$  be a compact 1-connected Lie group, then  $K^*(B_G)$  and  $K^*(G)$  are well-known, [8] and [31]. If the rank of  $G$  is  $r$ ,  $K^*(B_G)$  is a power series ring on  $r$  generators in  $K^0(B_G)$ , and  $K^*(G)$  is a primitively generated exterior algebra on  $r$  generators in  $K^1(G)$ . These results are in marked contrast to those for integral cohomology.  $H^*(B_G)$  and  $H^*(G)$  may have 2,3 or 5-torsion, depending on which group we are considering. It is a natural conjecture that  $K^*(\Omega G)$  should show a similar uniformity as compared to  $H^*(\Omega G)$ , [14]. In fact a little thought shows that the expected result would be that  $K^*(\Omega G)$  is a divided power series algebra on  $r$  generators in  $K^0(\Omega G)$ , or dually that  $K_*(\Omega G)$  is a polynomial algebra. It is the purpose of this thesis to demonstrate that this conjecture is false. We obtain results showing that  $K_*(\Omega G)$  has in general a strictly more complex Hopf algebra structure than  $H_*(\Omega G)$ , (3.6.8), (3.8.11).

An obvious method that might be used to prove the conjecture would be to pass from  $K^*(G)$  to  $K^*(\Omega G)$  by means of an Eilenberg-Moore spectral sequence for K-theory. A candidate for such a spectral sequence has been

constructed by L. Hodgkin, [33]. In the first chapter, we prove that this spectral sequence does not converge in the situation under consideration if  $H^*(G)$  has torsion. The method of proof uses only two properties of the spectral sequence, both of which are very natural generalisations of properties of the classical Eilenberg-Moore spectral sequence. Thus our proof extends to cover the various other spectral sequences of this type which have been constructed, for example in [46]. The question of convergence when  $H^*(G)$  is torsion-free is left open; the results we obtain later are consistent with convergence.

The second and third chapters are concerned with explicitly computing  $K_*(\Omega G)$  for various Lie groups. Of course it is no restriction to assume  $G$  is simply-connected, and by the decomposition theorem for Lie groups we may assume  $G$  is also simple.

In chapter two, we generalise to  $K$ -theory the method Bott used for finding  $H_*(\Omega G)$ , [14]. This leads to an algebraic expression for  $K^*(\Omega G)$  in terms of the representation theory of  $G$ . As far as actual calculations



are concerned, we are able to obtain the Hopf algebra structure of  $K_*(\Omega G)$  for  $G = \mathrm{SU}(n+1)$  and  $\mathrm{Sp}(2)$ .

The approach of chapter three is to work through complex bordism, and then apply the Conner-Floyd isomorphism. We use a method due to T. Petrie, [40], to determine the ring structure of  $\Omega_*^U(\Omega G)$  and extend it to deal with the diagonal structure. The idea is to notice that the homology generators of Bott, which come from Morse theory, are given by complex manifolds, to use these as bordism generators, and employ characteristic numbers to calculate the relations between them. We are able to obtain results for a wider class of groups than in chapter two. We end the chapter by considering the example of the exceptional Lie group  $G_2$ , and seeing how we can obtain Steenrod operations in  $H^*(\Omega G_2; \mathbb{Z}_p)$  quite easily from our calculations.

The work for this thesis was done in the years 1969-1971 at the University of Warwick. I would like to thank my two supervisors during this time, Luke Hodgkin and Alan Robinson for their advice and encouragement. My thanks are due to the Science Research Council for their grant.

### Notation

$\mathbb{Z}$ ,  $\mathbb{Z}_p$ ,  $\mathbb{Q}$ , and  $\mathbb{C}$  denote the integers, the integers modulo  $p$ , the rationals, and the complex numbers.

If  $R$  is a ring  $\Lambda_R(\ )$ ,  $R[\ ]$ , and  $R[[\ ]]$  denote respectively exterior algebra, a polynomial algebra, and a power series algebra over  $R$  on the elements within the brackets.

If  $X$  is a topological space, and  $R$  a ring,  $H^*(X;R)$  and  $H_*(X;R)$  denote the  $\mathbb{Z}$ -graded singular cohomology and homology groups of  $X$  with coefficients in  $R$ . We write  $H^{**}(X;R)$  for the direct product of the  $H^i(X;R)$ , and let  $H^*(X) = H^*(X;\mathbb{Z})$ ,  $H_*(X) = H_*(X;\mathbb{Z})$ .  $K^*(X)$  and  $K_*(X)$  are the representable  $K$ -theory generalised cohomology and homology groups, both considered as  $\mathbb{Z}_2$ -graded, [28] §2.

If  $X$  is a CW-complex  $\underline{K}^*(X) = \varprojlim_{\alpha} \{K^*(X_{\alpha})\}$ , where the limit is over all finite subcomplexes  $X_{\alpha}$ .  $X^p$  denotes the  $p$ -skeleton of  $X$ ; if all the skeletons are finite  $\underline{K}^*(X)$  is equal to  $\varprojlim_p \{K^*(X^p)\}$ .  $ch: K^*(X) \longrightarrow H^{**}(X;\mathbb{Q})$  and  $ch: K_*(X) \longrightarrow H_*(X;\mathbb{Q})$  are the Chern character homomorphisms

The pairing between two dual modules is written  $\langle \ , \ \rangle$ . When they are generalised cohomology and homology modules, the cohomology element is written on the left.

If  $X$  is a space with base point  $*$ ,  $\Lambda X$  is the space of paths on  $X$  at  $*$ ,  $\Omega X$  is the space of loops at  $*$ , and  $SX$  is the reduced suspension.  $I$  is the unit interval  $[0,1]$ , with base point at  $0$ , and  $S^n$  is the  $n$ -sphere.

If  $G$  is a Lie group,  $W(G)$  is its Weyl group,  $R(G)$  its representation ring, and  $B_G$  its classifying space. If  $H \subset G$  is a closed subgroup  $G/H$  is the space of cosets. The standard notations are used for the classical groups.

The numbering systems used has the following form.

(a.b.c) indicates the  $c$ -th item in §a.b of chapter a.

## Chapter 1

The main purpose of this chapter is to prove Theorem (1.4.1). This shows that the path space fibration on a Lie group with torsion provides an example for which the K-theory generalised Eilenberg-MacLane spectral sequence of Luke Hodgkin, [33], fails to converge. In the first section we describe the spectral sequence; in the second we identify the first filtration terms. Section 3 is devoted to recalling a Chern character result of B. Harris, and in the last section the theorem is proved.

### §1.1 The Hodgkin Spectral Sequence

In this section we describe the construction of a spectral sequence due to Luke Hodgkin [33].

Let  $B$  be a fixed topological space. Consider the category  $\text{Top}/B$  of spaces over  $B$ , whose objects are continuous maps  $p_X: X \rightarrow B$  and of which the morphisms are commuting diagrams

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ p_X \searrow & & \swarrow p_Y \\ & B & \end{array} .$$

The corresponding based category is denoted by  $\text{Top}/B_0$ . Its objects are maps  $p_X: X \rightarrow B$ , together with a cross-section  $k_X: B \rightarrow X$  satisfying  $p_X \circ k_X = 1_B$ . Its morphisms are maps  $X \rightarrow Y$  commuting both with  $p_X, p_Y$

and with  $k_X, k_Y$ .

For  $B = \text{point}$ , we obtain the categories Top of topological spaces and continuous maps, and Top<sub>0</sub> the usual based category.  $B$  with  $p_B = k_B = 1_B$  is an initial object in Top/B<sub>0</sub>.

Definition (1.1.1)

If  $f: X \rightarrow Y$  is a morphism in Top/B<sub>0</sub>, the mapping-cone of  $f$  is a space  $C_B(f)$  in Top/B<sub>0</sub>, defined as the disjoint union  $Y \cup X \times I \cup B$  with identifications:  $f(x) = (x, 1)$ ,  $(x, 0) = p_X(x)$ ,  $(k_X(b), t) = b$ . The maps from  $Y, X$ , and  $B$  to  $B$  combine to give  $C_B(f) \rightarrow B$ , and the cross-section is given by the inclusion of  $B$  in the union.

There is a natural map  $j(f): Y \rightarrow C_B(f)$  in Top/B<sub>0</sub> induced from the inclusion.

Definition (1.1.2)

The suspension of  $X$  in Top/B<sub>0</sub> is the mapping-cone of  $p_X: X \rightarrow B$ . It is written  $S_B(X)$ .

Definition (1.1.3)

The fibred product of  $X, Y$  in Top/B<sub>0</sub> is defined by

$$X \times_B Y = \{(x, y) \in X \times Y : p_X(x) = p_Y(y)\}$$

with  $p(x, y) = p_X(x) = p_Y(y)$ , and  $k(b) = (k_X(b), k_Y(b))$ .

All the usual constructions of homotopy theory go

through in  $\underline{\text{Top}}/B_0$ , see [46], [34]. In particular we have the Puppe sequence

$$X \xrightarrow{f} Y \xrightarrow{j(f)} C_B(f) \xrightarrow{l(f)} S_B(X) \xrightarrow{S_B(f)} S_B(Y) \longrightarrow \dots$$

Definition (1.1.4)

Define functors  $F: \underline{\text{Top}}/B_0 \longrightarrow \underline{\text{Top}}_0$ ,

$$G: \underline{\text{Top}}_0 \longrightarrow \underline{\text{Top}}/B_0$$

by  $F(X) = X/k_X B$  with the natural base point,

$G(Y) = Y \times B$  with  $p_{Y \times B} =$  projection onto  $B$ , and

$k_{Y \times B}(b) = (*, b)$ , where  $*$   $\in Y$  is the base point.

$F$  and  $G$  are adjoint functors with adjunction

$i: I \longrightarrow GF$  given by  $i(X): X \longrightarrow GF(X) = (X/k_X B) \times B$

$$x \longmapsto (x, p_X(x)).$$

Proposition (1.1.5)

1) On fibres the sequence  $X \xrightarrow{f} Y \xrightarrow{j(f)} C_B(f)$  is just the usual mapping-cone.

2) Applying the functor  $F$  to the same sequence we find that  $FC_B(f)$  is the mapping-cone of  $F(f)$ .

Proof: Immediate from the definitions.

Definition (1.1.6)

Let  $X$  be an object of  $\underline{\text{Top}}/B$ , then the cobar construction of  $X$  is the diagram in  $\underline{\text{Top}}/B_0$

$$X_0 \xrightarrow{i_0} Z_0 \xrightarrow{j_0} X_1 \xrightarrow{i_1} \dots \longrightarrow X_p \xrightarrow{i_p} Z_p \xrightarrow{j_p} X_{p+1} \longrightarrow \dots$$

defined inductively by

$$X_0 = X \cup B \text{ with } k_{X_0} \text{ the inclusion of } B,$$

$$Z_p = GF(X_p) = (X_p/kB) \times B,$$

$$i_p = i(X_p): X_p \longrightarrow Z_p.$$

$$x \longmapsto (x, px), \text{ see (1.1.4),}$$

$$X_{p+1} = C_B(i_p),$$

$$\text{and } j_p = j(i_p): Z_p \longrightarrow C_B(i_p) = X_{p+1}, \text{ see (1.1.1).}$$

Let  $\underline{W}$  be the full subcategory of  $\underline{Top}$  consisting of spaces having the homotopy type of a CW-complex, so that we have the homotopy extension property.

For  $B$  in  $\underline{W}$  we have categories  $\underline{W/B}$ ,  $\underline{W/B}_0$ ,  $\underline{W}_0$ .

If  $X \xrightarrow{f} Y$  is in  $\underline{W/B}_0$ ,  $X \longrightarrow Y \longrightarrow C_B(f)$  is a cofibration sequence, [46]. In particular, in the cobar construction for each  $p$   $X_p \longrightarrow Z_p \longrightarrow X_{p+1}$  is a cofibration, and the Puppe sequence

$$X_p \longrightarrow Z_p \longrightarrow X_{p+1} \longrightarrow S_B X_p \longrightarrow \dots$$

enables us to consider  $X_{p+1} \subset S_B X_p$ , and so in general  $X_{p+q} \subset S_B^q X_p$ .

Let  $h^*$  be a representable cohomology theory on  $\underline{W}$ .

#### Definition (1.1.7)

If  $X, Y$  are in  $\underline{W/B}$  the cobar construction spectral sequence  $\{E_r^p(X; Y)_B\}$ , relative to  $h^*$ , is the spectral sequence arising from the following  $H(p, q)$ -system

$$\begin{aligned}
H(-p, -q) &= h^*(S_B^{p-q} X_{q+1} \times Y, X_{p+1} \times Y) & -1 \leq q \leq p < \infty, \\
&= h^*(S_B^{p+1} X_0 \times Y, X_{p+1} \times Y) & -\infty \leq q < -1 \leq p < \infty, \\
&= 0 & -\infty \leq q \leq p < -1.
\end{aligned}$$

The maps are defined by restrictions and coboundaries and

$$H(-\infty, -q) = \varinjlim_p H(-p, -q) \quad -\infty \leq q < \infty,$$

$$\text{in particular } H = H(-\infty, \infty) = \varinjlim_p h^*(S_B^{p+1} X_0 \times Y, X_{p+1} \times Y).$$

Theorem (1.1.8) (Hodgkin)

If  $h^*$  is a multiplicative cohomology theory,  $Y$  has the homotopy type of a CW-complex with a countable number of cells, and  $h^*(X)$ ,  $h^*(B)$  are projective  $h^*$ -modules of finite type, then

1) The spectral sequence converges strongly to  $H$ .

2)  $E_1(X; B)_B$  is the graded bar construction of  $h^*(X)$  over  $h^*(B)$ , as a complex.

$$3) E_1(X; Y)_B \cong E_1(X; B)_B \otimes_{h^*(B)} h^*(Y).$$

$$4) E_2^{-p}(X; Y)_B = \text{Tor}_{h^*(B)}^{-p}(h^*(X), h^*(Y)).$$

Proof: [33] 3.3 and 4.5.

Note: We shall ignore throughout the secondary grading of the spectral sequence induced from the grading on  $h^*$ .

The inclusion  $(S_B^{p+1} X_0, B) \subset (S_B^{p+1} X_0, X_{p+1})$  induces

$$h^*(S_B^{p+1} X_0 \times Y, X_{p+1} \times Y) \longrightarrow h^*(S_B^{p+1} X_0 \times Y, B \times Y)$$

||| suspension isomorphism

$$h^*(X_0 \times Y, B \times Y)_B \cong h^*(X \times Y)_B,$$



which provides in the limit a homomorphism

$$\lambda: H \longrightarrow h^*(X \times_Y B) \quad (1.1.9)$$

If the spectral sequence is to be a useful generalisation of the Eilenberg-Moore spectral sequence,  $\lambda$  should be an isomorphism. The main theorem of this chapter, (1.4.1) exhibits a class of examples for which  $\lambda$  is not an isomorphism.

Proposition (1.1.10)

Under the assumptions of theorem (1.1.8) the spectral sequence has a natural multiplicative structure, such that the differentials are derivations, on the  $E_2$ -term the product in Tor is the usual one, and that  $\lambda$  of (1.1.9) is a ring homomorphism.

Proof: See [32] p49, also [46] §5.

§1.2 The filtration terms

The limit of the spectral sequence  $H$ , carries a filtration  $0 \subset F^0 H \subset F^{-1} H \subset \dots \subset F^{-p+1} H \subset F^{-p} H \subset \dots$  the corresponding graded system of which is the  $E_\infty$ -term of the spectral sequence.

The map  $\lambda: H \longrightarrow h^*(X \times_Y B)$  of (1.1.9) then induces a filtration on  $h^*(X \times_Y B)$ .

We shall investigate the first two terms of this filtration, in the case that  $Y$  is a point  $b' \in B$ , so that  $X \times_B Y = F$  the fibre of  $X$  over  $b'$ , and then that  $X = \Lambda B$  the space of paths on  $B$ , so that  $F = \Omega B$  the loop space of  $B$ .

Now, in terms of the  $H(p,q)$ -system, see [20],  
 $H = \varinjlim_p H(-p, \infty)$  and the filtration on  $H$  is given by  
 $F^{-p}H = \text{Im}\{H(-p, \infty) \rightarrow H\}.$

So the filtration on  $h^*(X \times_B Y)$  is given by

$$F^{-p}h^*(X \times_B Y) = \text{Im}\{H(-p, \infty) \rightarrow H \xrightarrow{\lambda} h^*(X \times_B Y)\}.$$

Now, see (1.1.7),

$$\begin{aligned} H(-p, \infty) &= h^*(S_B^{p+1} X_{0B} \times Y, X_{p+1B} \times Y) & -1 \leq p < \infty, \\ &= 0 & p < -1. \end{aligned}$$

Thus  $F^{-p}h^*(X \times Y) = 0$ ,  $p < -1$ .

**Proposition (1.2.1)**

If  $Y = \{b'\} \subset B$ , so that  $X \times_B Y = p_X^{-1}\{b'\} = F$ , then  
 $F^0 h^*(F) = \text{Im}\{i^*: h^*(X) \rightarrow h^*(F)\}$ , where  $i$  is the inclusion of the fibre.

**Proof:** The cobar construction provides us with a cofibration

$$(1.2.2) \quad X_0 \rightarrow Z_1 \rightarrow X_1 \quad \text{in } \underline{W/B}_0.$$

Applying  $\times_{\underline{B}}\{b'\}$  to this sequence, by Proposition (1.1.5), 1), we obtain a cofibration

$$(1.2.3) \quad F_0 \xrightarrow{i_0} (X_0/kB) \longrightarrow F_1 \quad \text{in } \underline{W}_0.$$

Here and below  $F_p$  denotes  $X_p \times_{PB} \{b'\}$ , the fibre of  $X_p$ , and  $Z_p = (X_p/kB) \times B$  so that the fibre of  $Z_p$  is  $(X_p/kB)$ .

Now  $X_0 = X \cup B$ , so  $X_0/kB = X^+$ ,  $X$  with disjoint base point. Thus  $F_0 = F^+$ , and the map  $i_0: F_0 \longrightarrow (X_0/kB)$  is the inclusion  $i: F \longrightarrow X$  and sends base point to base point. We may extend (1.2.3) to a Puppe sequence

$$(1.2.4) \quad F_0 \xrightarrow{i_0} X^+ \longrightarrow F_1 \longrightarrow SF_0 \xrightarrow{Si_0} S(X^+).$$

$$\text{Now } F^0 h^*(F) = \text{Im}\{h^*(SF_0, F_1) \longrightarrow \tilde{h}^*(SF_0) \cong \tilde{h}^*(F_0) \cong h^*(F)\}$$

Then (1.2.4) tells us that  $(SF_0/F_1) \simeq S(X^+)$ , so

$$\begin{aligned} F^0 h^*(F) &= \text{Im}\{(Si_0)^*: \tilde{h}^*(S(X^+)) \longrightarrow \tilde{h}^*(SF_0)\}, \\ &= \text{Im}\{i_0^*: \tilde{h}^*(X^+) \longrightarrow \tilde{h}^*(F_0)\}, \\ &= \text{Im}\{i^*: h^*(X) \longrightarrow h^*(F)\}. \end{aligned}$$

This proves the proposition.

We now specialise to the case when  $X = \Lambda B$  and  $F = \Omega B$ .

$X$  is contractible to  $x' = i(f')$ , where  $f'$  is the trivial loop at  $b'$ . Thus the inclusion  $i: F \longrightarrow X$  is homotopy split by  $X \xrightarrow{q} \{x'\} \subset F$ .  $F^0 h^*(F) = \text{Im}\{i^*: h^* \longrightarrow h^*(F)\}$  so

$$\frac{h^*(F)}{F^0 h^*(F)} = \text{Ker}\{q^*: h^*(F) \longrightarrow h^*\} = \tilde{h}^*(F).$$

Definition (1.2.5)

The cohomology suspension  $\sigma: \tilde{h}^*(B) \longrightarrow \tilde{h}^*(F)$  is defined

as follows. The identity map on  $F = \Omega B$  has adjoint

$g: SF = S\Omega B \longrightarrow B$ .  $\sigma$  is the composition

$$\tilde{h}^*(B) \xrightarrow{g^*} \tilde{h}^*(SF) \cong \tilde{h}^*(F).$$

Proposition (1.2.6)

If  $B$  is 1-connected then

$$\frac{F^{-1}h^*(F)}{F^0h^*(F)} \subset \frac{h^*(F)}{F^0h^*(F)} = \tilde{h}^*(F)$$

is the image of the cohomology suspension  $\sigma: \tilde{h}^*(B) \longrightarrow \tilde{h}^*(F)$ .

Proof: Applying the functor  $F$  of (1.1.4) to the sequence (1.2.2), we obtain

$$X_0/kB \longrightarrow Z_0/kB \longrightarrow X_1/kB$$

which is a cofibration in  $\underline{W}_0$  by Proposition (1.1.5), 2).

It is clearly equal to

$$\begin{array}{lcl} X^+ & \longrightarrow & (X \times B)^+ \longrightarrow X_1/kB, \\ x & \longmapsto & (x, px) \end{array}$$

which is homotopy equivalent to

$$\begin{array}{lcl} \{0, 1\} = S^0 & \longrightarrow & B^+ \longrightarrow X_1/kB. \\ 0 & \longmapsto & * \\ 1 & \longmapsto & b' \end{array}$$

So  $X_1/kB$  is homotopy equivalent to  $B$  with base point  $b'$ .

On the other hand as mapping-cone it is given as

$$\begin{aligned} X_1/kB = & \underline{X \times B \cup X \times I \cup \{*\}} \\ & (x, px) \sim (x, 1) \\ & (x, 0) \sim * \end{aligned}$$

We use here and below an obvious notation for identifications

Lemma (1.2.7)

A suitable homotopy equivalence with  $B$  is provided by

the map  $\varphi: X_1/kB \longrightarrow B$   
 $(x, b) \longmapsto b,$   
 $(x, t) \longmapsto x(t),$  considering  $x \in X = \Lambda B$

as map  $I \longrightarrow B$  so that  $x(1) = px,$

and  $* \longmapsto b'$

Proof: It is immediate to check that  $\varphi$  is well-defined.

Define  $\psi: B \longrightarrow X_1/kB$  by  $b \longmapsto (x', b) \in X \times B \subset X_1/kB$ . Now

$\varphi \circ \psi = \text{identity on } B$ . We show that  $\psi \circ \varphi$  is homotopic to the identity on  $X_1/kB$ .

Let  $G: X \times I \longrightarrow X$  be a homotopy between the identity and the collapsing maps which contracts  $X$  along paths, so that

$$G(x, s)(t) = x(st).$$

Define a map  $H: X_1/kB \times I = \underline{X \times B \times I \cup X \times I \times I \cup I} \longrightarrow X_1/kB$  by  
 $(x, px, s) \sim (x, 1, s)$   
 $(x, 0, s) \sim s$

on  $X \times B \times I$   $H: (x, b, s) \longmapsto (G(x, s), b) \in X \times B,$

on  $X \times I \times I$   $H: (x, t, s) \longmapsto \begin{cases} (G(x, st), 1+t-s) \in X \times I & 0 \leq t \leq s \leq 1, \\ (G(x, st), x(t-s+ts)) \in X \times B & 0 \leq s \leq t \leq 1, \end{cases}$

on  $I$   $H: s \longmapsto (x', 1-s) \in X \times I.$

It is easily checked that these agree on the identifications

and that  $H|_{X_1/kB \times \{0\}} = \psi \circ \varphi$ , whereas

$$\begin{aligned} H|_{X_1/kB \times \{1\}} &= \chi: (x, b) \longmapsto (x, b) \\ &\quad (x, t) \longmapsto (G(x, t), t) \\ &\quad * \longmapsto (x', 0) = *. \end{aligned}$$

So  $H: \psi \circ \varphi \simeq \chi$ . A homotopy  $K: \chi \simeq \text{identity}$  is given by

$$\text{on } X \times B \times I \quad K: (x, b, s) \longmapsto (x, b),$$

$$\text{on } X \times I \times I \quad K: (x, t, s) \longmapsto (G(x, s+t-st), t),$$

$$\text{on } I \quad K: s \longmapsto *.$$

Therefore  $\psi \circ \varphi \simeq \text{identity}$  and the lemma is proved.

We return to the proof of (1.2.6) and note that the

$$\begin{array}{ccccc} \text{diagram} & X^+ & \longrightarrow & (X \times B)^+ & \longrightarrow & X_1/kB \\ & \parallel & & \parallel & & \psi \parallel \varphi \\ & S^0 & \longrightarrow & B^+ & \longrightarrow & B \end{array} \quad \text{homotopy commutes.}$$

Now  $F_1 = \text{fibre of } X_1$  is given as  $\frac{X \cup F \times I \cup \{*\}}{\substack{\text{if } \sim(f, 1) \\ (f, 0) \sim *}}$ , the

inclusion  $i_1: F_1 \rightarrow X_1$  being the natural one, so that the composition  $F_1 \xrightarrow{i_1} X_1 \rightarrow X_1/kB \xrightarrow{\varphi} B$  is the map  $x \mapsto b', (f, t) \mapsto f(t), * \mapsto b'$ .

The sequence (1.2.4) becomes when  $X$  is contractible

$$F_0 \rightarrow S^0 \rightarrow F_1 \rightarrow SF_0 \rightarrow \dots \quad (1.2.8).$$

Then there is a cofibration  $I \rightarrow F_1 \rightarrow SF$ , where the suspension  $SF$  is relative to the base point  $f' \in F$ , in which the map  $I \rightarrow F_1 = \frac{\{*\} \cup F \times I \cup \{*\}}{\substack{* \sim (f, 1) \\ (f, 0) \sim *}}$  is  $t \mapsto (f', t)$ .

So the map  $F_1 \rightarrow X_1/kB \simeq B$  factors

$$\begin{array}{ccc} F_1 & \xrightarrow{\quad} & B \\ & \searrow & \uparrow g \\ & SF & \end{array}$$

where  $g$ , as in (1.2.5), is the adjoint of the identity on  $F$ .

The cofibration  $X_1 \longrightarrow Z_1 \longrightarrow X_2$  in  $\underline{W/B}_0$  yields, on taking fibres a cofibration  $F_1 \longrightarrow X_1/kB = B \longrightarrow F_2$ , and so a Puppe sequence  $F_1 \longrightarrow B \longrightarrow F_2 \longrightarrow SF_1 \longrightarrow SB \longrightarrow \dots$ , where, by what we have seen above, the first map is equivalent to  $g$ .

In the  $h^*$ -exact sequence of the triple  $(S^2F_0, SF_1, F_2)$  the coboundary  $h^*(SF_1, F_2) \cong \tilde{h}^*(SB) \longrightarrow \tilde{h}^*(S^2) \cong h^*(S^2F_0, SF_1)$  is induced from a map  $S^2 \longrightarrow SB$ , but  $\pi_2(SB) = 0$  since  $B$  is 1-connected, thus the exact sequence breaks up into a short exact sequence. The exact sequence of the cofibration  $SF_1 \longrightarrow S^2F_0 \longrightarrow S^2$  also breaks up, for similar reasons, and we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & h^*(S^2F_0, SF_1) & \longrightarrow & h^*(S^2F_0, F_2) & \longrightarrow & h^*(SF_1, F_2) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & h^*(S^2F_0, SF_1) & \longrightarrow & \tilde{h}^*(S^2F_0) & \longrightarrow & \tilde{h}^*(SF_1) \longrightarrow 0 \end{array}$$

which is isomorphic to

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{h}^*(S^2) & \longrightarrow & h^*(S^2F_0, F_2) & \longrightarrow & \tilde{h}^*(SB) \longrightarrow 0 \\ & & \parallel & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \tilde{h}^*(S^2) & \xrightarrow{\alpha} & \tilde{h}^*(S^2F_0) & \longrightarrow & \tilde{h}^*(SF_1) \longrightarrow 0 \end{array}$$

Now, up to suspension,  $F^0h^*(F)$  is identified with  $\text{Im}\alpha$ , and  $F^{-1}h^*(F)$  is identified with  $\text{Im}\beta$ . Therefore

$$\frac{F^{-1}h^*(F)}{F^0h^*(F)} \text{ is identified with } \text{Im}\gamma \subset \tilde{h}^*(SF_1) \cong \frac{h^*(F)}{F^0h^*(F)}.$$

Now  $\gamma$  is induced from the suspension of the map  $F_1 \longrightarrow E$  which we considered above, so we have a commutative diagram

$$\begin{array}{ccccc} \tilde{h}^*(F) & \cong & \tilde{h}^*(SF) & \xrightarrow{\cong} & \tilde{h}^*(F_1) \\ & \swarrow \sigma & \uparrow g^* & \nearrow \gamma & \\ & & \tilde{h}^*(B) & & \end{array} \quad (\text{desuspended})$$

Hence the proposition follows.

Now in the spectral sequence, see [20],

$$E_r^{-p} = Z_r^{-p} / B_r^{-p}, \text{ where}$$

$$Z_r^{-p} = \text{Im}\{H(-p, -p+r) \longrightarrow H(-p, -p+1)\},$$

$$B_r^{-p} = \text{Im}\{H(-p-r+1, -p) \xrightarrow{\delta} H(-p, p+1)\}.$$

So, taking  $p = 1$ , see (1.1.7),

$$Z_1^{-1} = h^*(SF_1, F_2),$$

$$Z_r^{-1} = \text{Im}\{h^*(S^2F_0, F_2) \longrightarrow h^*(SF_1, F_2)\}, \quad 2 \leq r.$$

But we saw in the proof of (1.2.6) that there is a short exact sequence

$$0 \longrightarrow h^*(S^2F_0, SF_1) \longrightarrow h^*(S^2F_0, F_2) \longrightarrow h^*(SF_1, F_2) \longrightarrow 0,$$

thus  $Z_r^{-1} = Z_1^{-1} = h^*(SF_1, F_2)$  for all  $r$  and

$$E_r^{-1} = \text{Coker}\{h^*(S^{r-1}F_2, F_{r+1}) \longrightarrow h^*(SF_1, F_2)\}.$$

There is then a sequence of homomorphisms

$$E_1^{-1} \longrightarrow E_2^{-1} \longrightarrow \dots \longrightarrow E_r^{-1} \longrightarrow E_{r+1}^{-1} \longrightarrow \dots \longrightarrow E_\infty^{-1} = \frac{F_H^{-1}}{F_H^0}.$$

Corollary (1.2.8)

The edge homomorphism



$$E_1^{-1} = h^*(SF_1, F_2) \cong \tilde{h}^*(B) \longrightarrow E_\infty^{-1} = \frac{F^{-1}H}{F^0H} \xrightarrow{\lambda} \tilde{h}^*(F)$$

is the cohomology suspension.

Proof: Follows immediately from the proof of Proposition (1.2.6).

Note: Under the identification of the  $E_2$ - term, theorem (1.1.8),  $E_1^{-1} \longrightarrow E_2^{-1} = \text{Tor}_{h^*(B)}^{-1}(h^*, h^*)$  is the natural map  $\tilde{h}^*(B) \longrightarrow Qh^*(B)$  - the indecomposable module of  $h^*(B)$ , through which the cohomology suspension factors.

The results of this section are exactly similar to those in the parallel case of the classical Eilenberg-Moore spectral sequence, [45]. It may be possible to prove analogues of the classical results for the general situation of arbitrary  $X, Y$  in  $\underline{W/B}$ , but we shall not need these in the sequel.

### §1.3 The Chern character on $K^*(\Omega G)$

Let  $G$  be a compact, 1-connected, simple Lie group, and  $\Omega G$  its loop space. Since  $G$  is simple,  $G$  is 2-connected and  $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$ , [10], generated by  $x_3$  say. The integral cohomology Serre spectral sequence of the path space fibration on  $G$  shows that  $H^2(\Omega G; \mathbb{Z}) \cong \mathbb{Z}$  generated by an element  $u_2$  such that the transgression  $\tau =$  the differential  $d_3$  maps

$u_2$  into  $x_3$ . Hence the cohomology suspension  $\sigma$  maps  $x_3$  into  $u_2$ , [43] Ch I,  $n^0_3$ .

We recall two theorems.

Theorem (1.3.1) (Hodgkin)

$$K^*(G) \cong \Lambda_{\mathbb{Z}}(\beta_{\rho_1}, \dots, \beta_{\rho_r}), \quad r = \text{rank } G,$$

where the  $\beta_{\rho_i} \in K^1(G)$  are primitive generators defined via the representation theory of  $G$ .

Proof: [31], or [6] and [3].

Theorem (1.3.2) (Harris)

Consider the generator  $x_3 \in H^3(G; \mathbb{Z})$  as an element of  $H^3(G; \mathbb{Q})$ , then

$$\text{ch } \beta_{\rho_i} = n_i x_3 + \{\text{higher terms}\} \in H^*(G; \mathbb{Q}),$$

where the  $n_i$  are integers, for which there is an algebraic expression in terms of the root system of  $G$ .

Let  $d = \text{hcf}(n_1, \dots, n_r)$ , then  $H^*(G; \mathbb{Z})$  has  $p$ -torsion if and only if  $p$  divides  $d$ .

Proof: [27].

Corollary (1.3.3)

Let  $\sigma_i = \sigma \beta_{\rho_i} \in \tilde{K}^*(\Omega G)$ , where  $\sigma$  is the K-theory cohomology suspension, (1.2.5), then

$$\text{ch } \sigma_i = n_i u_2 + \{\text{higher terms}\} \in H^{**}(\Omega G; \mathbb{Q}).$$

Proof: The diagram

$$\begin{array}{ccc} \tilde{K}^*(G) & \xrightarrow{\text{ch}} & \tilde{H}^*(G; \mathbb{Q}) \\ \downarrow \sigma & & \downarrow \sigma \\ \tilde{K}^*(\Omega G) & \xrightarrow{\text{ch}} & \tilde{H}^{**}(\Omega G; \mathbb{Q}) \end{array} \quad \text{commutes,}$$

since the Chern character is natural and commutes with coboundaries, [8]. The result follows from theorem (1.3.2) and the remark that  $\sigma x_3 = u_2$ .

Note that  $\{\sigma_1, \dots, \sigma_r\}$  is a basis for

$$\text{Im}\{\sigma: K^*(G) \longrightarrow K^*(\Omega G)\}$$

since  $\sigma$  maps decomposable elements to zero.

Proposition (1.3.4)

There is an element  $\alpha \in \tilde{K}^*(\Omega G)$  with

$$\text{ch}\alpha = u_2 + \{\text{higher terms}\}.$$

Proof: Direct application of the Corollary of 2.5 of [8].

This result extends to the infinite dimensional  $\Omega G$  since it depends only on the Atiyah-Hirzebruch spectral sequence for  $\Omega G$ . The spectral sequence collapses,  $H^*(\Omega G)$  being even-dimensional, [13], and so, by Proposition I,2.1 of [31], converges to  $\underline{K}^*(\Omega G)$ . This last is then isomorphic to  $K^*(\Omega G)$ , by Theorem 1' of [17].

#### §1.4 Non-convergence Theorem

In this section we prove a theorem showing non-

convergence, in that the map  $\lambda$  of (1.1.9) fails to be an isomorphism, of the spectral sequence of (1.1.7) for K-theory and the path space fibration on a Lie group with torsion. This provides a negative answer to the conjecture of [33], which was that the spectral sequence should converge when the base space had the homotopy type of a finite CW-complex.

We form the spectral sequence of (1.1.7) with

$$h^* = K^* \text{ representable K-theory,}$$

$$B = G \text{ a compact, 1-connected, simple Lie group,}$$

$$X = \Lambda G \text{ the space of paths on } G \text{ at } e \in G,$$

$$\text{and } Y = \{e\}, \text{ so that}$$

$$\underset{B}{X \times Y} = F = \Omega G.$$

Then we have  $\lambda: H \longrightarrow K^*(\Omega G)$ .

Theorem (1.4.1)

If  $H^*(G)$  has torsion,  $\lambda$  is not an epimorphism.

Note:  $\lambda$  is a monomorphism, since tensoring with the rationals the Chern character provides an isomorphism of the theories  $K^*(\cdot) \otimes \mathbb{Q}$  and  $H^*(\cdot; \mathbb{Q})$  and  $\lambda$  is an isomorphism for ordinary theories, as is shown in [33]. On the other hand we shall see in the proof of the theorem that  $H$  is torsion-free.

Proof: The  $E_2$ -term of the spectral sequence is, (1.1.8),

$$\begin{aligned} E_2^{-p} &= \text{Tor}_{K^*(G)}^{-p}(K^*(\Lambda G), K^*({e})), \\ &= \text{Tor}_{K^*(G)}^{-p}(\mathbb{Z}, \mathbb{Z}). \end{aligned}$$

Since  $K^*(G)$  is an exterior algebra on  $r$  ( $= \text{rank } G$ ) generators, (1.3.1),  $E_2^*$  is a divided polynomial algebra on  $r$  generators in dimension  $-1$ , see [19].

So  $E_2^{-p} = 0$  for  $p < 0$ , but  $d_r^{-p}: E_r^{-p} \rightarrow E_r^{-p+r}$ , thus  $d_r^{-1} = 0$  for  $r \geq 2$ . By (1.1.10), the differentials are derivations, hence the spectral sequence collapses and  $E_\infty^* = E_2^*$ .

Now the filtration

$$0 \subset F^0 H \subset F^{-1} H \subset \dots \subset F^{-p} H \subset F^{-p-1} H \subset \dots$$

on  $H$  is such that  $\frac{F^{-p} H}{F^{-p-1} H} \cong E_\infty^{-p} = E_2^{-p}$ . So that

$$F^0 H \cong E_2^0 \cong \mathbb{Z}, \text{ generated by } 1.$$

Now  $E_2^{-1} = \text{Tor}_{K^*(G)}^{-1}(\mathbb{Z}, \mathbb{Z}) \cong \text{QK}^*(G)$  the indecomposable module of  $K^*(G)$ , and (1.2.8) shows us that composed with  $\lambda$  the above identification is essentially the cohomology suspension.

Let  $x_1, \dots, x_r \in E_2^{-1}$  be generators corresponding to  $\{\beta_{e_1}\}, \dots, \{\beta_{e_r}\}$ , where  $\{ \}$  denotes the class in the quotient  $\text{QK}^*(G)$ . If  $y_1, \dots, y_r$  are elements of  $F^{-1} H$  giving corresponding generators of  $\frac{F^{-1} H}{F^0 H}$ , then

$$\lambda y_i = \sigma_i \in \tilde{K}^*(\Omega G), \quad i = 1, \dots, r.$$

Now  $E_2^{-p}$  has a basis of elements of the form

$$\frac{(x_1)^{k_1}}{k_1!} \dots \frac{(x_r)^{k_r}}{k_r!} \quad k_1 + \dots + k_r = p, \quad k_i \geq 0.$$

It follows, by induction using (1.1.10), that  $F^{-p}H$  has a basis of elements

$$1, \quad \frac{(y_1)^{k_1}}{k_1!} \dots \frac{(y_r)^{k_r}}{k_r!} \quad 0 < k_1 + \dots + k_r \leq p, \quad k_i \geq 0.$$

Let  $\alpha \in K^*(\Omega G)$  be the element provided by (1.3.4) with

$$\text{ch}\alpha = u_2 + \{\text{higher terms}\},$$

suppose there is an element  $y \in F^{-p}H$  which  $\lambda$  maps into  $\alpha$ .

We can write

$$y = \text{Sum of elements of the form } \frac{(y_1)^{k_1}}{k_1!} \dots \frac{(y_r)^{k_r}}{k_r!},$$

$$\lambda y = \alpha = \text{Sum of elements of the form } \frac{(\sigma_1)^{k_1}}{k_1!} \dots \frac{(\sigma_r)^{k_r}}{k_r!},$$

$$= a_1 \sigma_1 + \dots + a_r \sigma_r + \{\text{elements of higher degree}\},$$

where the  $a_i$  are integers.

$$\text{Now } \text{ch}\left(\frac{(\sigma_1)^{k_1}}{k_1!} \dots \frac{(\sigma_r)^{k_r}}{k_r!}\right) = \frac{n_1^{k_1} \dots n_r^{k_r}}{k_1! \dots k_r!} u_2^{k_1 + \dots + k_r} \\ + \{\text{higher terms}\},$$

see (1.3.3). Thus

$$\text{ch}\alpha = (a_1 n_1 + \dots + a_r n_r) + \{\text{higher terms}\},$$

and so  $a_1 n_1 + \dots + a_r n_r = 1$ . But, since  $H^*(G)$  has torsion,

(1.3.2) tells us that the  $n_i$  have highest common factor

greater than 1, so we have a contradiction and

$$\alpha \notin \text{Im}\{\lambda: F^{-p}H \longrightarrow K^*(\Omega G)\}.$$

By Cartan-Eilenberg's axiom (S.P.5), [20],  $H = \bigcup_p F^{-p}H$ , therefore  $\alpha \notin \text{Im}\{\lambda: H \longrightarrow K^*(\Omega G)\}$ , and the theorem is proved.

Since  $\Omega G$  is an infinite dimensional space  $K^*(\Omega G)$  can be expected to have a non-trivial topology with respect to the skeleton filtration of [8]. We might then only expect  $\lambda: H \longrightarrow K^*(\Omega G)$  to map onto a dense subspace, but we have the following

Corollary (1.4.2)

$\text{Im}\lambda \subset K^*(\Omega G)$  is not dense with respect to the skeleton filtration topology.

Proof: We have seen from the proof of the theorem that there is an element  $\alpha \in K^*(\Omega G)$  such that  $\text{ch}\alpha = u_2 + \{\text{higher terms}\}$  which is not in the image of  $\lambda$ . Since by [8], see also the proof of (1.3.4), the Chern character defines the filtration topology, there is a neighbourhood  $V$  of  $\alpha$  in  $K^*(\Omega G)$  such that  $\text{ch}\beta = u_2 + \{\text{higher terms}\}$  for all  $\beta \in V$ . Then the proof of the theorem shows us that  $V \cap \text{Im}\lambda = \emptyset$ .

If  $G$  is a semi-simple, 1-connected, compact Lie group we can write  $G = G_1 \times \dots \times G_k$ , where each  $G_i$  is simple and 1-connected, [12]. By the Künneth formula  $G$  has torsion in its cohomology if and only if one of the  $G_i$ 's does, say  $G_1$ . Put  $G_2 \times \dots \times G_k = G'$ , then

$$K^*(G) \cong K^*(G_1) \otimes K^*(G'), \quad [5],$$

and the Chern character on  $K^*(G)$  is the tensor product of the Chern characters on  $K^*(G_1)$  and  $K^*(G')$ . It is easy to see that the results of §1.3 can be extended, we have

$$\begin{aligned} H^*(\Omega G; \mathbb{Q}) &= H^*(\Omega G_1; \mathbb{Q}) \otimes H^*(\Omega G'; \mathbb{Q}), \\ &= \mathbb{Q}[u_2, \dots] \otimes H^*(\Omega G'; \mathbb{Q}), \end{aligned}$$

and  $\text{Im}\{\sigma: \hat{K}^*(\Omega G) \rightarrow \hat{K}^*(\Omega G)\}$  has a basis  $\{\sigma_1, \dots, \sigma_{r_1}, \dots, \sigma_r\}$ ,  $r_1 = \text{rank } G_1$ ,  $r = \text{rank } G$ , such that

$$\begin{aligned} \text{ch } \sigma_i &= n_i u_2 + \{\text{higher terms}\}, \quad i = 1, \dots, r_1 \\ \text{and } \text{ch } \sigma_i &\in 1 \otimes H^*(\Omega G'; \mathbb{Q}), \quad \text{for } i = r_1 + 1, \dots, r. \end{aligned}$$

We also have an  $\alpha \in K^*(\Omega G)$  with  $\text{ch } \alpha = u_2 + \{\text{higher terms}\}$ .

Since  $H^*(G_1)$  has torsion the  $n_i$  have highest common factor greater than 1, and we have enough machinery to make the proof of the theorem work for

### Corollary (1.4.3)

Theorem (1.4.1) holds for  $G$  semi-simple.



We note that the proof of the theorem used only the identification of the edge-homomorphism of (1.2.8) and the multiplicative structure of the spectral sequence. These two properties are very natural generalisations of the corresponding results for the classical Eilenberg-Moore spectral sequence, [46], and so we can conjecture that there is no generalised Eilenberg-Moore spectral sequence for K-theory which converges for the path space fibration on a Lie group with torsion.

## Chapter 2

In this chapter we extend the method used by Bott in [14] to obtain the homology of the space of loops on a Lie group to K-theory. The first three sections are devoted to setting up the necessary algebraic machinery. In the last four we apply this to our topological situation. We show how the algebraic construction behaves well with respect to the Adams operations and obtain complete results for the Unitary groups and the first Symplectic group. We have not been able to make the more complicated calculations for other groups. In the last section we make use of a result on the K-theory of homogeneous spaces to prove a theorem, (2.7.8), which in principle, provides, together with the results of Bott in [14], a complete algebraic description of  $K^*(\Omega G)$  in terms of representation theory. We have not yet been able to make computational use of this result.

### §2.1. Completed Tensor Products and Inverse Limits

In this section we prove the technical results necessary to extend the ideas of §6 of [14] from the graded to the filtered case.

Let  $R$  be a fixed Noetherian commutative ring with unit, and let  $\otimes = \otimes_R$  throughout.

Definition (2.1.1)

An  $R$ -module  $M$  is filtered if we are given a decreasing sequence

$$M = M_0 \supset M_1 \supset \dots \supset M_p \supset M_{p+1} \supset \dots$$

of  $R$ -submodules.

A morphism  $f: M \rightarrow N$  of filtered modules is an  $R$ -homomorphism such that  $f(M_p) \subset N_p$ .

The completion of a filtered module  $M$  is defined by

$$M^\wedge = \varprojlim_p \left\{ \frac{M}{M_p} \right\},$$

there is a natural map  $M \rightarrow M^\wedge$  induced from the projections.

$M$  is complete if this map is an isomorphism.  $M^\wedge$  has a filtration defined by  $(M^\wedge)_p = \text{Ker}\{M^\wedge \rightarrow M/M_p\}$  with respect to which it is complete.

$M$  is freely filtered if for each  $p$ ,  $M_p$  and  $M/M_p$  are free  $R$ -modules.

$M$  is finitely filtered if for each  $p$ ,  $M/M_p$  is a finitely generated  $R$ -module.

$M$  is discretely filtered if for some  $n$ ,  $M_n = 0$ .

Definition (2.1.2)

Let  $M, N$  be two filtered  $R$ -modules. Filter the tensor

product  $M \otimes N$  by

$$(M \otimes N)_r = \sum_{p+q=r} M_p \otimes N_q.$$

The completed tensor product  $\hat{M} \hat{\otimes} N$  is the completion of  $M \otimes N$  with respect to this filtration. That is

$$\hat{M} \hat{\otimes} N = \varprojlim_r \left( \frac{M \otimes N}{\sum_{p+q=r} M_p \otimes N_q} \right)$$

Proposition (2.1.3)

If  $M$  is freely filtered then

$$\hat{M} \hat{\otimes} N = \varprojlim_{p,q} \left\{ \frac{M}{M_p} \otimes \frac{N}{N_q} \right\}.$$

Proof: For each  $p, q$ , since  $M_p$ ,  $M/M_p$  and hence  $M$  are free, we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_p \otimes N_q & \longrightarrow & M_p \otimes N & \longrightarrow & M_p \otimes N / N_q \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M \otimes N_q & \longrightarrow & M \otimes N & \longrightarrow & M \otimes N / N_q \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M/M_p \otimes N_q & \longrightarrow & M/M_p \otimes N & \longrightarrow & M/M_p \otimes N / N_q \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where all the rows and columns are exact. Hence there is a short exact sequence

$$0 \longrightarrow M \otimes N_q + M_p \otimes N \longrightarrow M \otimes N \longrightarrow M/M_p \otimes N / N_q \longrightarrow 0,$$

obtained by a simple diagram chase, see [36] II.5 Ex3.

Therefore

$$\varprojlim_{p,q} \left\{ \frac{M}{M_p} \otimes \frac{N}{N_q} \right\} = \varprojlim_{p,q} \left\{ \frac{M \otimes N}{M \otimes N_q + M_p \otimes N} \right\} = \varprojlim_r \left\{ \frac{M \otimes N}{M \otimes N_r + M_r \otimes N} \right\},$$

the diagonal subsystem being cofinal. But

$$(M \otimes N)_{2r} \subset M_r \otimes N + M \otimes N_r \subset (M \otimes N)_r.$$

Thus the two filtrations of  $M \otimes N$ , by  $(M \otimes N)_r$  and by  $(M_r \otimes N + M \otimes N_r)$ , define the same topology on  $M \otimes N$ , so the completions are isomorphic.

Proposition (2.1.4)

Let  $M, N, P$  be three filtered  $R$ -modules with  $M, P$  freely filtered then  $(M \hat{\otimes} N) \hat{\otimes} P = M \hat{\otimes} (N \hat{\otimes} P)$ .

Proof: By a simple application of Lemma (3.11) of [4], the map  $M \hat{\otimes} N \rightarrow \frac{M}{M_r} \otimes \frac{N}{N_r}$  is an epimorphism and the proof

of (2.1.3) shows that kernels of such maps provide a filtration on  $M \hat{\otimes} N$  equivalent to that determined by (2.1.1) and (2.1.2). So by (2.1.3) again

$$(M \hat{\otimes} N) \hat{\otimes} P = \varprojlim_r \left\{ \left( \frac{M}{M_r} \otimes \frac{N}{N_r} \right) \otimes \frac{P}{P_r} \right\}.$$

Now use the result on ordinary tensor products.

Lemma (2.1.5)

Let  $\{U_i\}$  be an inverse system of  $R$ -modules, indexed over the non-negative integers, and  $V$  a free

finitely generated R-module, then the natural map

$$V \otimes \varprojlim_i \{U_i\} \longrightarrow \varprojlim_i \{V \otimes U_i\}$$

is an isomorphism.

Proof: See [41] Theorem 2.

Proposition (2.1.6)

If  $M$  is finitely generated and freely filtered then

$$M \hat{\otimes} N = M^\wedge \otimes N^\wedge.$$

Proof: It follows immediately from the assumptions on  $M$

that there is an integer  $n_0$  such that  $M_n = M_{n_0}$  for all  $n \geq n_0$ .

So that  $M^\wedge = M/M_{n_0}$  is free and finitely generated.

$$\text{Hence } M \hat{\otimes} N = \varprojlim_{p,q} \{M/M_p \otimes N/N_q\}, \text{ by (2.1.3),}$$

$$= \varprojlim_q \{M^\wedge \otimes N/N_q\},$$

$$= M^\wedge \otimes N^\wedge, \text{ by (2.1.5).}$$

Proposition (2.1.7)

Let  $\{M^i\}_{i \geq 0}$  be an inverse system of filtered modules such that

- 1) Each  $M^i$  is finitely generated and discrete,
- 2) For each  $p$  the inverse system  $\{M_p^i\}_{i \geq 0}$  satisfies the condition (ML) of [4] §3.

Write  $M = \varprojlim_i \{M^i\}$ , filtered by  $M_p = \varprojlim_i \{M_p^i\}$ , and

let  $N$  be a freely and finitely filtered module. Then

$$M \hat{\otimes} N = \varprojlim_i \{M^i \otimes N^\wedge\}.$$

Proof: By assumption 2), the exact sequences

$$0 \longrightarrow M_p^i \longrightarrow M^i \longrightarrow M^i/M_p^i \longrightarrow 0,$$

give in the limit, see [4] (3.8), an exact sequence

$$0 \longrightarrow M_p \longrightarrow M \longrightarrow \varprojlim_i \{M^i/M_p^i\} \longrightarrow 0,$$

showing that  $M/M_p = \varprojlim_i \{M^i/M_p^i\}$ .

$$\begin{aligned} \text{Now } M \hat{\otimes} N &= \varprojlim_{p,q} \{M/M_p \otimes N/N_q\}, \text{ by (2.1.3),} \\ &= \varprojlim_{p,q} \{(\varprojlim_i \{M^i/M_p^i\}) \otimes N/N_q\}, \\ &= \varprojlim_{p,q,i} \{M^i/M_p^i \otimes N/N_q\}, \text{ by (2.1.5),} \\ &= \varprojlim_i \{M^i \otimes N\}, \\ &= \varprojlim_i \{M^i \otimes N^\wedge\}, \text{ by (2.1.6).} \end{aligned}$$

Proposition (2.1.8)

Let  $\{M^i\}_{i \geq 0}$ ,  $\{N^j\}_{j \geq 0}$  be two inverse systems of complete, freely filtered, finitely generated modules, both satisfying condition 2) of Proposition (2.1.7).

Let  $M = \varprojlim_i \{M^i\}$ ,  $N = \varprojlim_j \{N^j\}$ , filtered as in (2.1.7).

$$\text{Then } M \hat{\otimes} N = \varprojlim_{i,j} \{M^i \otimes N^j\}.$$

Proof: As in the proof of (2.1.7),  $M/M_p = \varprojlim_i \{M^i/M_p^i\}$ ,

hence  $M^\wedge = \varprojlim_{p,i} \{M^i/M_p^i\} = \varprojlim_i \{M^{i^\wedge}\} = M$ , and similarly for  $N$ .

Thus, by (2.1.7),

$$\begin{aligned} M \hat{\otimes} N &= \varprojlim_i \{M^i \hat{\otimes} N\}, \\ &= \varprojlim_i \{\varprojlim_j \{M^i \otimes N^j\}\}, \end{aligned}$$

by another application of (2.1.7), which proves the proposition.

We see now how this construction applies to K-theory.

Let  $X$  be a CW-complex, then by  $K^*(X)$  we mean the representable K-theory of  $X$ . We can filter  $K^*(X)$  by

$$K^*(X)_p = \text{Ker}\{K^*(X) \longrightarrow K^*(X^{p-1})\},$$

where  $X^{p-1}$  is the  $(p-1)$ -skeleton. Recall from [4] that this filtration is an invariant of homotopy type.

Suppose  $X$  and  $Y$  are spaces with the homotopy type of CW-complexes, then  $X \times Y$  is also such a space. The exterior product

$$K^*(X) \otimes K^*(Y) \longrightarrow K^*(X \times Y)$$

then maps  $K^*(X)_p \otimes K^*(Y)_q$  into  $K^*(X \times Y)_{p+q}$ , [42] 5.3.

So with respect to the tensor product filtration of (2.1.2) the exterior product is a filtered map.



Proposition (2.1.9)

Let  $X$  and  $Y$  be spaces with the homotopy type of CW-complexes with finite skeletons, and such that their integral cohomology is torsion free, then the exterior product factors to give an isomorphism

$$K^*(X) \hat{\otimes} K^*(Y) \longrightarrow K^*(X \times Y)$$

of filtered  $\mathbb{Z}$ -modules.

Proof: Since  $H^*(X)$ ,  $H^*(Y)$  and hence  $H^*(X \times Y)$  are torsion-free of finite type, the Atiyah-Hirzebruch spectral sequences for the three spaces collapse, [8]. They strongly converge therefore to  $\underline{K}^*(X)$ ,  $\underline{K}^*(Y)$ ,  $\underline{K}^*(X \times Y)$ , and these are equal to  $K^*(X)$ ,  $K^*(Y)$ ,  $K^*(X \times Y)$ , respectively, [17]. Thus  $K^*(X)$ ,  $K^*(Y)$ ,  $K^*(X \times Y)$  are all complete freely and finitely filtered  $\mathbb{Z}$ -modules, [4].

The exterior product induces a map of the spectral sequences

$$E_r^p(X) \otimes E_r^q(Y) \longrightarrow E_r^{p+q}(X \times Y),$$

which on the  $E_2$ -terms is the isomorphism induced by the cohomology exterior product, and on the  $E_\infty$ -terms is compatible with the K-theory exterior product.

Thus in the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & (K^*(X) \otimes K^*(Y))_r & \longrightarrow & K^*(X) \otimes K^*(Y) & \longrightarrow & \frac{K^*(X) \otimes K^*(Y)}{(K^*(X) \otimes K^*(Y))_r} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow f & \\
0 \longrightarrow & K^*(X \times Y)_r & \longrightarrow & K^*(X \times Y) & \longrightarrow & \frac{K^*(X \times Y)}{K^*(X \times Y)_r} & \longrightarrow 0
\end{array}$$

the map  $f$  is identified with the isomorphism

$$\sum_{p+q \leq r} H^p(X) \otimes H^q(Y) \longrightarrow \sum_{i \leq r} H^i(X \times Y).$$

In the limit therefore we have

$$\begin{array}{ccc}
K^*(X) \otimes K^*(Y) & \longrightarrow & K^*(X) \hat{\otimes} K^*(Y) \\
\downarrow & & \downarrow \cong \\
K^*(X \times Y) & \xrightarrow{\cong} & K^*(X \times Y)^\wedge
\end{array}$$

The fact that the isomorphism is filtered follows immediately from the proof.

Note: This result is a special case of an infinite dimensional Künneth formula of V.M. Buhštaber, Theorem 4.1 of [18].

Proposition (2.1.10)

If  $X$  is an H-space, and  $H^*(X)$  is torsion-free of finite type, then the maps

$$\begin{aligned}
\varphi: K^*(X) \hat{\otimes} K^*(X) &\longrightarrow K^*(X \times X) \xrightarrow{\Delta^*} K^*(X), \\
\psi: K^*(X) &\xrightarrow{m^*} K^*(X \times X) \xleftarrow{\cong} K^*(X) \hat{\otimes} K^*(X),
\end{aligned}$$

where  $m$  and  $\Delta$  are the multiplication and diagonal maps,

together with the augmentation and unit

$$\epsilon: K^*(X) \longrightarrow \mathbb{Z}, \quad \eta: \mathbb{Z} \longrightarrow K^*(X),$$

induced from the natural maps  $\text{point} \subset X$  as the identity, and  $X \rightarrow \text{point}$ , give  $K^*(X)$  the structure of a  $\mathbb{Z}_2$ -graded completed Hopf algebra. That is the structure diagrams of Milnor and Moore, [39], all commute with  $\hat{\otimes}$  replacing  $\otimes$  throughout.

Proof:  $K^*(\text{point}) = \mathbb{Z}$  has the filtration  $K^*(\text{point})_p = 0$ ,  $p \geq 1$ . Hence by (2.1.6)

$$K^*(X) \hat{\otimes} \mathbb{Z} = K^*(X)^\wedge \otimes \mathbb{Z} = K^*(X)^\wedge = K^*(X).$$

It is trivial then that the diagrams of the form

$$\begin{array}{ccc} K^*(X) \hat{\otimes} \mathbb{Z} & \xrightarrow{1 \hat{\otimes} \eta} & K^*(X) \hat{\otimes} K^*(X) \\ & \searrow & \swarrow \varphi \\ & K^*(X) & \end{array} \quad \text{commute.}$$

Associativity of the product and diagonal follow immediately from their definitions and Proposition (2.1.4).

It remains to show commutativity of the diagram

$$\begin{array}{ccccc} K^*(X) \hat{\otimes} K^*(X) & \xrightarrow{\varphi} & K^*(X) & \xrightarrow{\psi} & K^*(X) \hat{\otimes} K^*(X) \\ \psi \hat{\otimes} \psi \downarrow & & & & \uparrow \varphi \hat{\otimes} \varphi \\ K^*(X) \hat{\otimes} K^*(X) \hat{\otimes} K^*(X) \hat{\otimes} K^*(X) & \xrightarrow{1 \hat{\otimes} T \hat{\otimes} 1} & K^*(X) \hat{\otimes} K^*(X) \hat{\otimes} K^*(X) \hat{\otimes} K^*(X) & & \end{array}$$

where  $T$  is the twisting homomorphism induced from

$$a \otimes b \longmapsto (-1)^{\dim a \cdot \dim b} b \otimes a.$$

But, by (2.1.10), (2.1.4), and [52] §6, this diagram is

$$\begin{array}{ccccc} \text{isomorphic to} & K^*(X \times X) & \xrightarrow{\Delta^*} & K^*(X) & \xrightarrow{m^*} & K^*(X \times X) \\ & \downarrow (m \times m)^* & & & & \uparrow (\Delta \times \Delta)^* \\ & K^*(X \times X \times X \times X) & \xrightarrow{(1 \times t \times 1)^*} & K^*(X \times X \times X \times X) & & \end{array}$$

where  $t: X \times X \rightarrow X \times X$  is the map  $(x, y) \mapsto (y, x)$ . That the corresponding diagram of spaces commutes is easy to verify, hence the result.

### Corollary (2.1.11)

Under the hypotheses of (2.1.10),  $K^*(X)$  is a filtered Hopf algebra, that is  $\varphi$  and  $\psi$  are filtered maps, where  $K^*(X) \hat{\otimes} K^*(X)$  is given the filtration of (2.1.2).

Proof: Follows trivially from the definitions.

## §2.2 Infinite Symmetric Powers

Let  $A$  be a free finitely generated  $R$ -module, with an augmentation  $\varepsilon: A \rightarrow R$  and unit  $\eta: R \rightarrow A$  such that  $\eta \cdot \varepsilon = 1_R$ . Define inductively  $A^1 = A$ ,  $A^n = A^{n-1} \otimes A$  the  $n$ -fold tensor product of  $A$ . Define  $\varepsilon^n: A^n \rightarrow A^{n-1}$  as

$$A^n = A^{n-1} \otimes A \xrightarrow{1 \otimes \varepsilon} A^{n-1} \otimes R = A^{n-1},$$

that is  $\varepsilon^n(a_1 \otimes \dots \otimes a_n) = \varepsilon(a_n) a_1 \otimes \dots \otimes a_{n-1}$ . (2.2.1)

Definition (2.2.2)

$G_n$ , the symmetric group on  $n$  symbols, acts in the obvious way on  $A^n$  by permuting the order of the tensor product.

Define  $\mathcal{S}^n A$  to be the submodule of elements of  $A^n$  fixed by the action of  $G_n$ .

Proposition (2.2.3)

$\mathcal{S}^n A$  is a free finitely generated  $R$ -module, and a direct summand of  $A^n$ .

Proof: If  $A$  has a basis  $\{a_0, a_1, \dots, a_k\}$ , which we may choose such that  $a_0 = 1 = \eta(1)$ ,  $\varepsilon(a_i) = 0$   $i \geq 1$ , then  $A^n$  has basis  $S = \{a_{i_1} \otimes \dots \otimes a_{i_n} : i_j \in \{0, 1, \dots, k\}\}$ . The action of  $G_n$  on  $A^n$  is determined by its action on  $S$ . Let  $S = S_1 \cup \dots \cup S_m$  be the decomposition of  $S$  into  $G_n$ -orbits. Pick  $s_i \in S_i$ ,  $i = 1, \dots, m$ , then  $\mathcal{S}^n A$  has a basis  $\{[s_i] : i = 1, \dots, m\}$ , where  $[s_i] = \sum_{s \in S_i} s$ , thus

$\mathcal{S}^n A$  is free.

Define a map  $A^n \rightarrow \mathcal{S}^n A$  such that on the basis  $S$

$$s_i \mapsto [s_i],$$

$$s \mapsto 0 \quad \text{if } s \neq s_i \text{ for some } i.$$

Then this splits the inclusion  $\mathcal{S}^n A \subset A^n$ , showing that

$\mathcal{S}^n A$  is a direct summand.

Proposition (2.2.4)

$\varepsilon^n$  maps  $\mathcal{S}^n A$  into  $\mathcal{S}^{n-1} A$ .

Proof: If  $u \in A^n$ ,  $\sigma \in G_{n-1} \subset G_n$ , then (2.2.1) shows that

$$\varepsilon^n(\sigma u) = \sigma \varepsilon^n(u).$$

The proposition follows.

By Proposition (2.2.4), we may form an inverse system

$$R \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon^2} \mathcal{S}^2 A \xleftarrow{\quad} \dots \xleftarrow{\quad} \mathcal{S}^{n-1} A \xleftarrow{\varepsilon^n} \mathcal{S}^n A \xleftarrow{\quad} \dots \quad (2.2.5)$$

Definition (2.2.6)

The infinite symmetric power of  $A$ , written  $\mathcal{S}A$ , is the inverse limit of (2.2.5). The maps of the inverse system provide a homomorphism  $\varepsilon^*: \mathcal{S}A \longrightarrow A$ , and an augmentation  $\varepsilon: \mathcal{S}A \longrightarrow R$ .

Suppose now that  $A$  is freely filtered, see (2.1.1),

$$A = A_0 \supset A_1 \supset \dots \supset A_p \supset A_{p+1} \supset \dots$$

and complete, hence that  $A_s = 0$  for some  $s$ , since  $A$  is finitely generated. Assume that  $A_1 = \text{Ker}\{\varepsilon: A \longrightarrow R\}$ .

Inductively  $A^n$  has the filtration

$$(A^n)_r = \sum_{p+q=r} (A^{n-1})_p \otimes A_q \subset A^{n-1} \otimes A = A^n,$$

and we can give  $\mathcal{S}^n A$  the filtration

$(\mathcal{S}^n A)_p = (A^n)_p \cap \mathcal{S}^n A = \mathcal{G}_n$ -invariant submodule of  $(A^n)_p$ .

Since  $\varepsilon$  maps  $A_1$  to zero  $\varepsilon^n: A^n \rightarrow A^{n-1}$  is a filtered homomorphism, and thus so is  $\varepsilon^n: \mathcal{S}^n A \rightarrow \mathcal{S}^{n-1} A$ , and we can give  $\mathcal{S}A$  the filtration  $(\mathcal{S}A)_p = \varprojlim_n \{(\mathcal{S}^n A)_p\}$ .

Proposition (2.2.7)

$$\text{For } n \geq p, \quad \frac{\mathcal{S}A}{(\mathcal{S}A)_p} \cong \frac{\mathcal{S}^{n-1} A}{(\mathcal{S}^{n-1} A)_p}.$$

Proof: Consider the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & (\mathcal{S}^{n-1} A)_p & \longrightarrow & \mathcal{S}^{n-1} A & \longrightarrow & \mathcal{S}^{n-1} A / (\mathcal{S}^{n-1} A)_p & \longrightarrow 0 \\ & \uparrow \varepsilon_p^n = \varepsilon^n | & & \uparrow \varepsilon^n & & \uparrow \tilde{\varepsilon}^n & \\ 0 \longrightarrow & (\mathcal{S}^n A)_p & \longrightarrow & \mathcal{S}^n A & \longrightarrow & \mathcal{S}^n A / (\mathcal{S}^n A)_p & \longrightarrow 0. \end{array}$$

From the diagram  $\tilde{\varepsilon}^n$  is an epimorphism since  $\varepsilon^n$  is; we will have proved that it is a monomorphism if we can show that, if  $x \in \mathcal{S}^n A$  is such that  $\varepsilon^n x \in (\mathcal{S}^{n-1} A)_p$ , then  $x \in (\mathcal{S}^n A)_p$ .

Case 1: If  $\varepsilon^n x = 0$ , then, with the notation of the proof of (2.2.3), write  $x = \sum r_{i_1 \dots i_n} a_{i_1} \otimes \dots \otimes a_{i_n}$ ,  $r_{i_1 \dots i_n} \in R$ ,

where the summation is over all  $i_1, \dots, i_n \in \{0, 1, \dots, k\}$ .

Now, since  $(a_0) = 1$ ,  $(a_i) = 0$   $i \geq 1$ ,

$$0 = \varepsilon^n x = \sum r_{i_1 \dots i_{n-1} 0} a_{i_1} \otimes \dots \otimes a_{i_{n-1}},$$

showing that  $r_{i_1 \dots i_{n-1} 0} = 0$ . Similarly by symmetry if

any  $i_j = 0$  then  $r_{i_1 \dots i_n} = 0$ . Therefore, in the expansion

of  $x$  above, all the non-zero summands are multiples of elements  $a_{i_1} \otimes \dots \otimes a_{i_n}$  with  $i_1, \dots, i_n \in \{1, \dots, k\}$ , that is

$a_{i_1}, \dots, a_{i_n} \in \text{Ker } \epsilon = A_1$ . Thus  $x \in (\mathcal{S}^n A)_n \subset (\mathcal{S}^n A)_p$ .

Case 2: In general. In the notation of the proof of (2.2.3),

let  $\{[s_i] : i=1, \dots, m\}$  be a basis for  $\mathcal{S}^{n-1} A$ , which we can

clearly choose so that  $\{[s_i] : i=1, \dots, l\}$  is a basis for

$(\mathcal{S}^{n-1} A)_p \subset \mathcal{S}^{n-1} A$ . Define  $f^n: \mathcal{S}^{n-1} A \rightarrow \mathcal{S}^n A$  by

$$[s_i] \mapsto [s_i \otimes 1] \quad 1 = \eta(1).$$

(recall  $[ ]$  indicates summation over the  $G_n$ -orbit), then

$\epsilon^n \cdot f^n = \text{identity}$ . So if  $x \in \mathcal{S}^n A$  and  $\epsilon^n x \in (\mathcal{S}^{n-1} A)_p$ , since

$\epsilon^n(x - f^n \epsilon^n x) = 0$ , by Case 1, it is sufficient to prove that

$f^n \epsilon^n x \in (\mathcal{S}^n A)_p$ . But by construction  $f^n$  maps  $(\mathcal{S}^{n-1} A)_p$  into

$(\mathcal{S}^n A)_p$ .

Now since  $\epsilon_p^n$  is an epimorphism the inverse system

$\{(\mathcal{S}^n A)_p\}_{n \geq 0}$  satisfies the condition (ML) of [4]. Hence

in the limit there is an exact sequence

$$0 \longrightarrow (\mathcal{S} A)_p \longrightarrow \mathcal{S} A \longrightarrow \varprojlim_n \{\mathcal{S}^n A / (\mathcal{S}^n A)_p\} \longrightarrow 0.$$

The result now follows.



Corollary (2.2.8)

$\mathcal{S}A$  is complete with respect to the filtration defined above.

Proof:

$$\begin{aligned}\mathcal{S}A^{\wedge} &= \varprojlim_p \{\mathcal{S}A/(\mathcal{S}A)_p\}, \\ &= \varprojlim_{p,n} \{\mathcal{S}^n A/(\mathcal{S}^n A)_p\}, \text{ by the proof of (2.2.7)} \\ &= \varprojlim_n \{\mathcal{S}^n A\}, \text{ since } \mathcal{S}^n A \text{ is complete} \\ &= \mathcal{S}A.\end{aligned}$$

We proceed now to show how  $\mathcal{S}A$  has a canonical completed coalgebra structure.

Definition (2.2.9)

For each  $i, j \geq 0$  we define

$$\Delta_{i,j}: A^{i+j} \longrightarrow A^i \otimes A^j$$

to be the isomorphism

$$a_1 \otimes \dots \otimes a_{i+j} \longmapsto (a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_{i+j}).$$

$\Delta_{i,j}$  factors to give

$$\Delta_{i,j}: \mathcal{S}^{i+j} A \longrightarrow \mathcal{S}^i A \otimes \mathcal{S}^j A,$$

which clearly maps  $(\mathcal{S}^{i+j} A)_p$  into  $(\mathcal{S}^i A \otimes \mathcal{S}^j A)_p$ .

Proposition (2.2.10)

The following two diagrams commute

$$\begin{array}{ccc}
S^{i+j}A & \xrightarrow{\Delta_{i,j}} & S^iA \otimes S^jA \\
\epsilon^{i+j} \downarrow & & \downarrow 1 \otimes \epsilon^j \\
S^{i+j-1}A & \xrightarrow{\Delta_{i,j-1}} & S^iA \otimes S^{j-1}A,
\end{array}$$

$$\begin{array}{ccc}
S^{i+j}A & \xrightarrow{\Delta_{i,j}} & S^iA \otimes S^jA \\
\epsilon^{i+j} \downarrow & & \downarrow \epsilon^i \otimes 1 \\
S^{i+j-1}A & \xrightarrow{\Delta_{i-1,j}} & S^{i-1}A \otimes S^jA.
\end{array}$$

Proof: That the first commutes is trivial since it does so on the  $A^{i+j}$ -level, that is if  $A^{i+j}$  replaces  $S^{i+j}A$  etc.

Commutativity of the second diagram follows if one notes that the homomorphisms  $\epsilon^n$  defined as in (2.2.1) and  $'\epsilon^n$  defined by  $'\epsilon^n(a_1 \otimes \dots \otimes a_n) = \epsilon(a_1)a_2 \otimes \dots \otimes a_n$  agree on the submodule  $S^nA$ .

If we consider the inverse system  $\{S^iA \otimes S^jA\}$  indexed over the ordered set  $\mathbb{Z}^+ \times \mathbb{Z}^+$ , with maps  $\epsilon^i \otimes 1, 1 \otimes \epsilon^j$ , Proposition (2.2.10) shows that, with respect to the map  $\mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+, (i,j) \mapsto i+j$ , of indexing sets, the  $\Delta_{i,j}$  form a morphism of inverse systems and we obtain in the limit

$$\varprojlim_{i,j} \{\Delta_{i,j}\} = \Delta: SA = \varprojlim_{i+j} \{S^{i+j}A\} \longrightarrow \varprojlim_{i,j} \{S^iA \otimes S^jA\} \quad (2.2.11)$$

Lemma (2.2.12)

$\varprojlim_{i,j} \{S^iA \otimes S^jA\}$  is naturally isomorphic to  $SA \hat{\otimes} SA$ .

Proof: To apply Proposition (2.1.8) we need that

$(S^n A)_p \subset S^n A$  is a direct summand for each  $n, p$ .

Since each  $A_p \subset A$  is a direct summand by assumption we may choose the basis  $\{a_0, \dots, a_k\}$  of  $A$  so that  $\{a_0, \dots, a_{k_r}\}$  is a basis for  $A_{s-r}$ ,  $r=1, \dots, s$  (recall that  $A_s = 0$ ). Then the basis  $\{a_{i_1} \otimes \dots \otimes a_{i_n} : i_j \in \{0, \dots, k\}\}$  for  $A^n$  is such that  $\{a_{i_1} \otimes \dots \otimes a_{i_n} : i_j \leq k_{r_j}, \sum_{j=1}^n r_j \leq ns - p\}$  forms a basis for  $(A^n)_p$ . Hence  $(A^n)_p$  is a direct summand of  $A^n$  and it follows that  $(S^n A)_p$  is of  $S^n A$ .

We may now apply (2.1.8) to obtain the required result.

Definition (2.2.13)

Define  $\eta^n: R \longrightarrow S^n A$  by  $1 \longmapsto 1 \otimes \dots \otimes 1$ , then  $\eta^n$  commutes with the maps of the inverse system (2.2.5) and determines  $\eta: R \longrightarrow SA$ .

Proposition (2.2.14)

The map  $\Delta: SA \longrightarrow SA \hat{\otimes} SA$  provided by (2.2.11) and (2.2.12) together with the homomorphisms  $\varepsilon, \eta$  of (2.2.6) and (2.2.13) give  $SA$  the structure of a completed, filtered, commutative coalgebra, with diagonal  $\Delta$ , unit  $\varepsilon$ , and augmentation  $\eta$ .

Proof: For  $i, j, k > 0$

$$\begin{array}{ccc}
 S^{i+j+k}_A & \xrightarrow{\Delta_{i,j+k}} & S^i_A \otimes S^{j+k}_A \\
 \Delta_{i+j,k} \downarrow & & \downarrow 1 \otimes \Delta_{j,k} \\
 S^{i+j}_A \otimes S^k_A & \xrightarrow{\Delta_{i,j} \otimes 1} & S^i_A \otimes S^j_A \otimes S^k_A
 \end{array}$$

commutes since it does so on the  $A^{i+j+k}$ -level. Then in the limit, using (2.1.4) and an obvious extension of (2.2.12), we obtain the commuting diagram

$$\begin{array}{ccc}
 S_A & \xrightarrow{\Delta} & S_A \hat{\otimes} S_A \\
 \Delta \downarrow & & \downarrow 1 \hat{\otimes} \Delta \\
 S_A \hat{\otimes} S_A & \xrightarrow{\Delta \hat{\otimes} 1} & S_A \hat{\otimes} S_A \hat{\otimes} S_A
 \end{array}$$

which shows that  $\Delta$  is associative.

Now  $\varepsilon: S_A \rightarrow R$  is defined, see (2.2.6), as the inverse limit of  $\{\varepsilon_n: S^n A \rightarrow R\}$ , where  $\varepsilon_n = \varepsilon \circ \dots \circ \varepsilon^{n-1} \circ \varepsilon^n$ . Thus the following commutes since it does so at the  $A^{i+j}$ -level,

$$\begin{array}{ccc}
 S^{i+j}_A & \xrightarrow{\varepsilon^{i+1} \circ \dots \circ \varepsilon^{i+j}} & S^i_A \\
 \Delta_{i,j} \downarrow & & \parallel \\
 S^i_A \otimes S^j_A & \xrightarrow{1 \otimes \varepsilon_j} & S^i_A \otimes R
 \end{array}$$

Then in the limit we have

$$\begin{array}{ccc}
 S_A & \xrightarrow{=} & S_A \\
 \Delta \downarrow & & \parallel \\
 S_A \hat{\otimes} S_A & \xrightarrow{1 \hat{\otimes} \varepsilon} & S_A \hat{\otimes} R, \text{ commuting, using (2.1.6)}
 \end{array}$$

The commutativity of the mirror image diagram, and of

those involving  $\eta$ , follow similarly using a similar trick to that employed in the proof of (2.2.10)

For commutativity of the coalgebra note that

$$\begin{array}{ccc} S^{i+j}A & \xrightarrow{\Delta_{i,j}} & S^iA \otimes S^jA \\ & \searrow \Delta_{j,i} & \downarrow T \\ & & S^jA \otimes S^iA \end{array}$$

commutes, where  $T$  is the twisting homomorphism, since if  $u \in A^{i+j}$   $T\Delta_{i,j}u = \Delta_{j,i}(\tau u)$  where  $\tau \in G_{i+j}$  is the element  $(12\dots i+j)^i$ . In the limit this provides the required diagram.

The fact that the coalgebra is filtered follows since  $\Delta_{i,j}$  is a filtered map, as noted in (2.2.9), and hence so is  $\Delta$ .

Definition (2.2.15)

Let  $\tilde{A} = \text{Ker}\epsilon = \text{Coker}\eta$ , and define

$$i_n: \tilde{A} \longrightarrow \tilde{S}^n A, \quad \tilde{S}^n A = \text{Ker}\{\epsilon_n: S^n A \longrightarrow R\},$$

$$\text{by } i_n(a) = a \otimes 1 \otimes \dots \otimes 1 + 1 \otimes a \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes a,$$

$$= [a \otimes 1 \otimes \dots \otimes 1], \text{ in the notation of the proof}$$

of (2.2.3), where  $1 = \eta(1) \in A$ .

Then  $\epsilon^n \cdot i_n = i_{n-1}$ , so we have a limit map

$$i: \tilde{A} \longrightarrow \tilde{S}A, \quad \tilde{S}A = \text{Ker}\{\epsilon: A \longrightarrow R\}.$$

Proposition (2.2.16)

$\text{Im}\{i: \tilde{A} \longrightarrow \tilde{S}A\} = \text{PS}A$ , the primitive module of the coalgebra  $S A$ .

Proof: By definition, [39],  $\text{PS}A = \text{Ker}\{\tilde{\Delta}: \tilde{S}A \longrightarrow \tilde{S}A \hat{\otimes} \tilde{S}A\}$ , where  $\tilde{\Delta}$  is defined as the composition

$$\tilde{S}A \subset S A \xrightarrow{\Delta} S A \hat{\otimes} S A \xrightarrow{\pi \hat{\otimes} \pi} \tilde{S}A \hat{\otimes} \tilde{S}A,$$

where  $\pi = (1 - \eta\epsilon): S A \longrightarrow \tilde{S}A$  is the projection.

Define  $\tilde{\Delta}_{i,j}$  as the composition

$$\widetilde{S^{i+j}A} \subset S^{i+j}A \xrightarrow{\Delta_{i,j}} S^i A \hat{\otimes} S^j A \xrightarrow{\pi_i \hat{\otimes} \pi_j} \tilde{S}^i A \hat{\otimes} \tilde{S}^j A.$$

We claim that

$$0 \longrightarrow \tilde{A} \xrightarrow{i_{i+j}} \widetilde{S^{i+j}A} \xrightarrow{\tilde{\Delta}_{i,j}} \tilde{S}^i A \hat{\otimes} \tilde{S}^j A \quad (2.2.17)$$

is exact.

It is easy to see that  $i_{i+j}$  is a monomorphism and that

$\tilde{\Delta}_{i,j} \circ i_{i+j} = 0$ . We prove by induction on  $i$  and  $j$  that

$$\text{Ker} \tilde{\Delta}_{i,j} \subset \text{Im} i_{i+j}.$$

For the basis of the induction consider the case

$i = j = 1$ . It is clear to see that  $\tilde{\Delta}_{1,1}$  is the composition

$$\widetilde{S^2 A} \subset A^2 = A \hat{\otimes} A \xrightarrow{\pi \hat{\otimes} \pi} \tilde{A} \hat{\otimes} \tilde{A} = \tilde{S}^1 A \hat{\otimes} \tilde{S}^1 A,$$

and hence that  $\text{Ker} \tilde{\Delta}_{1,1} = \widetilde{S^2 A} \cap \text{Ker}(\pi \hat{\otimes} \pi) = \text{Im} i_2$ .

For the induction step we consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{A} & \xrightarrow{i_{i+j}} & \widetilde{S^{i+j}A} & \xrightarrow{\tilde{\Delta}_{i,j}} & \tilde{S}^i A \hat{\otimes} \tilde{S}^j A \\ & & \parallel & & \uparrow \epsilon^{i+j+1} & & \uparrow \epsilon^{i+1} \otimes 1 \\ 0 & \longrightarrow & \tilde{A} & \xrightarrow{i_{i+j+1}} & \widetilde{S^{i+j+1}A} & \xrightarrow{\tilde{\Delta}_{i+1,j}} & \tilde{S}^{i+1} A \hat{\otimes} \tilde{S}^j A \end{array} \quad (2.2.18)$$

which commutes by (2.2.10). (The argument for the similar diagram obtained by replacing  $(i+1, j)$  by  $(i, j+1)$  in the bottom right hand corner is the same.)

Claim:  $\text{Ker } \varepsilon^{i+j+1} \cap \text{Ker } \tilde{\Delta}_{i+1, j} = 0$ .

Let  $x \in \text{Ker } \varepsilon^{i+j+1}$  then expressing  $x$  as a sum of basis elements in  $A^{i+j+1}$ , as in the proof of (2.2.7),  $x$  is a sum of elements of the form  $x_1 \otimes \dots \otimes x_{i+j+1}$  where  $x_\alpha \in \tilde{A}$   $\alpha=1, \dots, i+j+1$ . Then

$$\tilde{\Delta}_{i+1, j} x = x \in \widetilde{S^{i+1} A} \otimes \widetilde{S^j A} \subset A^{i+1} \otimes A^j \cong A^{i+j+1},$$

so if  $x \in \text{Ker } \tilde{\Delta}_{i+1, j}$ ,  $x = 0$ , proving the claim.

Now assume that the top row of (2.2.18) is exact, and suppose  $\tilde{\Delta}_{i+1, j} x = 0$ , then

$$\tilde{\Delta}_{i, j} \varepsilon^{i+j+1} x = (\varepsilon^{i+1} \otimes 1) \tilde{\Delta}_{i+1, j} x = 0.$$

Thus there is an element  $a \in \tilde{A}$  such that  $i_{i+j} a = \varepsilon^{i+j+1} x$ , then  $\varepsilon^{i+j+1} (i_{i+j+1} a - x) = 0$ , but  $\tilde{\Delta}_{i+1, j} \circ i_{i+j+1} = 0$ , as we noted above, hence  $i_{i+j+1} a - x \in \text{Ker } \tilde{\Delta}_{i+1, j} \cap \text{Ker } \varepsilon^{i+j+1} = 0$ , by the claim. So  $x = i_{i+j+1} a$ , and the bottom row is exact. This completes the induction.

In the limit we obtain, by [4] (3.8), an exact sequence

$$0 \longrightarrow \tilde{A} \xrightarrow{i} \tilde{S}A \xrightarrow{\tilde{\Delta}} \tilde{S}A \hat{\otimes} \tilde{S}A,$$

which proves the proposition.

Assume now that in addition  $A$  is a commutative algebra over  $R$ , with product  $m: A \otimes A \longrightarrow A$ , and for which  $\epsilon$  and  $\eta$  are augmentation and unit. We will assume also that  $A$  is a filtered algebra, that is

$$m: A_p \otimes A_q \longrightarrow A_{p+q}.$$

We give  $A^n$  the tensor product algebra structure, that is

$$m_n: A^n \otimes A^n \longrightarrow A^n$$

is defined by  $(a_1 \otimes \dots \otimes a_n) \otimes (b_1 \otimes \dots \otimes b_n) \mapsto m(a_1 \otimes b_1) \otimes \dots \otimes m(a_n \otimes b_n)$ .

$\mathcal{S}^n A$  is a filtered subalgebra, and  $\epsilon^n: \mathcal{S}^n A \longrightarrow \mathcal{S}^{n-1} A$  an algebra homomorphism. We obtain in the limit

$$m: \mathcal{S} A \hat{\otimes} \mathcal{S} A \longrightarrow \mathcal{S} A,$$

which gives  $\mathcal{S} A$  the structure of a completed, filtered, commutative algebra over  $R$ .

Proposition (2.2.19)

$\mathcal{S} A$  is a completed Hopf algebra over  $R$ .

Proof: All that remains to prove is that  $\Delta$  is an algebra homomorphism. The diagram

$$\begin{array}{ccc}
 \mathcal{S}^{i+j} A \otimes \mathcal{S}^{i+j} A & \xrightarrow{\Delta_{i,j} \otimes \Delta_{i,j}} & \mathcal{S}^i A \otimes \mathcal{S}^j A \otimes \mathcal{S}^i A \otimes \mathcal{S}^j A \\
 \downarrow m_{i+j} & & \downarrow 1 \otimes \tau \otimes 1 \\
 & & \mathcal{S}^i A \otimes \mathcal{S}^i A \otimes \mathcal{S}^j A \otimes \mathcal{S}^j A \\
 & & \downarrow m_i \otimes m_j \\
 \mathcal{S}^{i+j} A & \xrightarrow{\Delta_{i,j}} & \mathcal{S}^i A \otimes \mathcal{S}^j A
 \end{array}$$

commutes since it does so at the  $A^{i+j}$ -level. Then in the



limit we obtain the required diagram.

Definition (2.2.20)

Suppose  $H$  is a completed, filtered Hopf algebra over  $R$  with unit and augmentation, which has commutative coalgebra structure. Let  $f: H \rightarrow A$  be a morphism of augmented, filtered algebras with unit.

Let  $\Delta: H \rightarrow H \hat{\otimes} H$  be the diagonal of  $H$ , define inductively  $\hat{H}^1 = H$ ,  $\hat{H}^n = \hat{H}^{n-1} \hat{\otimes} H$ , and  $\Delta^n: \hat{H}^n \rightarrow \hat{H}^{n+1}$  by

$$\hat{H}^n = \hat{H}^{n-1} \hat{\otimes} H \xrightarrow{1 \hat{\otimes} \Delta} \hat{H}^{n-1} \hat{\otimes} H \hat{\otimes} H = \hat{H}^{n+1};$$

put  $\Delta_n = \Delta^{n-1} \circ \dots \circ \Delta: H \rightarrow \hat{H}^n$ .  $G_n$  acts on  $\hat{H}^n$  and, if  $\hat{S}^n H$  is the invariant submodule, by commutativity of  $\Delta$ ,  $\Delta_n$  maps into  $\hat{S}^n H$ . Clearly  $f$  induces

$$\hat{S}^n f: \hat{S}^n H \rightarrow \hat{S}^n A = S^n A, \text{ since } A \text{ is discrete.}$$

The maps  $\hat{S}^n f \cdot \Delta_n: H \rightarrow S^n A$  clearly commute with the  $\epsilon^n$ ,

so we can define

$$\beta f = \varprojlim_n \{\hat{S}^n f \cdot \Delta_n\}: H \rightarrow SA,$$

which will be a filtered map.

Proposition (2.2.21)

$Sf$  is a morphism of completed Hopf algebras such that the diagram

$$\begin{array}{ccc} & & SA \\ & \nearrow Sf & \downarrow \epsilon^* \\ H & \xrightarrow{f} & A \end{array} \quad \text{commutes.}$$

( $\epsilon^*$  is as in Definition (2.2.6).)

Proof: Since  $f$  is an algebra homomorphism so is  $\hat{S}^n f$ , and  $\Delta_n$  is an algebra homomorphism since  $\Delta$  is. Thus  $\hat{S}f$  is an algebra homomorphism.

For each  $i, j \geq 1$ , it is easily checked that the diagram

$$\begin{array}{ccccc}
 H \hat{\otimes} H & \xrightarrow{\Delta_i \hat{\otimes} \Delta_j} & \hat{S}^i H \hat{\otimes} \hat{S}^j H & \xrightarrow{\hat{S}^i f \hat{\otimes} \hat{S}^j f} & \hat{S}^i A \hat{\otimes} \hat{S}^j A \\
 \uparrow \Delta & & \uparrow \Delta_{i,j} & & \uparrow \Delta_{i,j} \\
 H & \xrightarrow{\Delta_{i+j}} & \hat{S}^{i+j} H & \xrightarrow{\hat{S}^{i+j} f} & \hat{S}^{i+j} A
 \end{array}$$

commutes, showing, on taking limits, that  $f$  is a coalgebra homomorphism.

That the diagram of the statement commutes is a consequence of the facts that  $f$  commutes with the augmentations, and that the augmentation in  $H$  is a unit for  $\Delta$ .

### Corollary (2.2.22)

If  $PH \subset \tilde{H}$  is the primitive module, the following diagram commutes

$$\begin{array}{ccc}
 & & SA \\
 & \nearrow Sf| & \uparrow i \\
 PH & \xrightarrow{f|} & A
 \end{array}$$

Proof: Immediate from the definitions of  $\hat{S}f$  and  $i$ .

We may apply the process of taking the symmetric power to a free, finitely generated, commutative algebra which is graded over the non-negative integers, and, if

the kernel of the augmentation is in strictly positive dimension, by taking graded tensor products there is no need to use completions. The infinite symmetric power will then be a graded Hopf algebra.

Proposition (2.2.23)

Let  $A$  be a filtered algebra as above, and let  $GA$  be the corresponding graded algebra, then

$$G(\mathfrak{S}A) \cong \mathfrak{S}(GA),$$

as graded Hopf algebras.

Proof: Trivial.

### §2.3 Infinite Symmetric Products and Duality

We begin this section by studying the construction dual to that of §2.2. This is more familiar; the proofs are dual, but easier due to the good behaviour of direct limits. Thus we do not need to use completed tensor products. We state most of the results without proof.

Let  $C$  be a free finitely generated  $R$ -module with unit and augmentation  $R \xrightleftharpoons[\varepsilon]{\eta} C$ . We assume that we have an increasing filtration

$$0 \subset C_0 \subset \dots \subset C_p \subset C_{p+1} \subset \dots \subset C_s = C,$$

for some  $s$ , such that  $C_p \subset C$  is a free direct summand

for all  $p$ , and  $C_0 = \text{Im } \eta$ .

Define  $C^1 = C$ ,  $C^{n+1} = C^n \otimes C$ , and  $\eta^n: C^n \longrightarrow C^{n+1}$  as

$$C^n = C^n \otimes R \xrightarrow{1 \otimes \eta} C^n \otimes C = C^{n+1}.$$

Inductively let

$$(C^n)_r = \sum_{p+q=r} (C^{n-1})_p \otimes C_q,$$

then this defines an increasing filtration on  $C^n$  having the same properties as that on  $C$ .

$G_n$  acts on  $C^n$ , preserving the filtration. Let

$U^n$  be the submodule generated by elements  $(u - \sigma \tilde{u})$ ,

$u \in C^n$ ,  $\sigma \in G_n$ , then define

$$S^n C = C^n / U^n.$$

$S^n C$  inherits an increasing filtration from  $C^n$ , and  $\eta^n$  factors to give  $\eta^n: S^n C \longrightarrow S^{n+1} C$ . We have a direct system of increasingly filtered  $R$ -modules

$$C \longrightarrow S^2 C \longrightarrow \dots \longrightarrow S^n C \xrightarrow{\eta^n} S^{n+1} C \longrightarrow \dots \quad (2.3.1)$$

Definition (2.3.2)

The infinite symmetric product of  $C$ , written  $SC$ , is defined as the direct limit of (2.3.1).  $SC$  has a filtration induced from those of the  $S^n C$ . We have induced maps

$$R \xrightleftharpoons[\epsilon]{\eta} SC \text{ and } \eta^*: C \longrightarrow SC.$$

Proposition (2.3.3)

$S^n C$  is a free finitely generated  $R$ -module.

Proof: Dual to that of Proposition (2.2.3).

Proposition (2.3.4)

For  $n \geq p$ ,  $(SC)_p \cong (S^n C)_p$ .

Proof: Dual to that of Proposition (2.2.7).

Corollary (2.3.5)

$SC = \varinjlim_p \{(SC)_p\}$ .

Proof: As Corollary (2.2.8)

The natural isomorphism  $C^i \otimes C^j \longrightarrow C^{i+j}$  factors to

$$m_{i,j}: S^i C \otimes S^j C \longrightarrow S^{i+j} C,$$

and in the limit  $m: SC \otimes SC \longrightarrow SC$ ,

making  $SC$  an algebra with unit  $\eta$  and augmentation  $\epsilon$ . It is a filtered algebra, in the sense that

$$m: (SC)_p \otimes (SC)_q \longrightarrow (SC)_{p+q}.$$

If  $C$  started off as a filtered coalgebra, then  $SC$  becomes a Hopf algebra. Given an increasingly filtered Hopf algebra  $K$ , and a map  $g: C \longrightarrow K$  which is a filtered coalgebra morphism, there is defined a filtered Hopf algebra homomorphism  $Sg: SC \longrightarrow K$  such that the diagram

$$\begin{array}{ccc} & SC & \\ \eta^* \uparrow & \searrow Sg & \\ C & \xrightarrow{\epsilon} & K \end{array}$$

(2.3.5)

commutes.

Suppose now that  $A, C$  are dual filtered  $R$ -modules,  
 that is  $A = \text{Hom}(C, R)$ , (We write  $\text{Hom} = \text{Hom}_R$ .)  
 and thus, since  $C$  is free and finitely generated

$$C = \text{Hom}(A, R),$$

and the filtrations are given by

$$A_p = \text{Annihilator of } C_{p-1} = \text{Hom}(C/C_{p-1}, R),$$

so that  $C_p = \text{Annihilator of } A_{p+1}$ .

Assume also that  $A \xrightleftharpoons[\eta]{\epsilon} R, R \xrightleftharpoons[\epsilon]{\eta} C$  are dual.

Proposition (2.3.6)

The inverse system (2.2.5) and the direct system  
 (2.3.1) of filtered modules are dual.

Proof: Consider the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \text{Hom}(S^n C, R) = \text{Annih}(U^n) & \longrightarrow & \text{Hom}(C^n, R) & \longrightarrow & \text{Hom}(U^n, R) & \longrightarrow & 0 \\ & & \parallel & & & & \\ 0 \longrightarrow S^n A & \longrightarrow & A^n & \longrightarrow & A^n / S^n A & \longrightarrow & 0. \end{array}$$

The rows are exact, the top one by (2.3.3), and the  
 isomorphism is provided by V.4.3 of [36].

Suppose  $a \in S^n A$ ,  $\sigma \in G_n$ , and  $(u - \sigma u) \in U^n$ , then

$$\begin{aligned} \langle a, u - \sigma u \rangle &= \langle a, u \rangle - \langle a, \sigma u \rangle, \\ &= \langle a, u \rangle - \langle \sigma^{-1} a, u \rangle, \\ &= 0, \text{ since } \sigma^{-1} a = a. \end{aligned}$$

Thus there is induced a monomorphism  $S^n A \longrightarrow \text{Hom}(S^n C, R)$ .

Suppose now that  $\lambda \in \text{Hom}(S^n C, R)$ , that is  $\lambda: C^n \rightarrow R$  and annihilates  $U^n$ . Considering  $\lambda$  as an element of  $A^n$ , so that  $G_n$  acts upon it, for  $\sigma \in G_n$ ,  $c \in C^n$

$$\begin{aligned} \langle \sigma \lambda, c \rangle &= \langle \lambda, \sigma^{-1} c \rangle, \\ &= \langle \lambda, \sigma^{-1} c \rangle + \langle \lambda, c - \sigma^{-1} c \rangle, \\ &= \langle \lambda, c \rangle. \end{aligned}$$

Thus  $\lambda \in S^n A$ , and this provides an inverse  $\text{Hom}(S^n C, R) \rightarrow S^n A$ . Since  $S^n C$  is free, (2.3.3),  $S^n C = \text{Hom}(S^n A, R)$ .

The duality of  $\epsilon^n$  and  $\eta^n$  is deduced from that of  $\epsilon$  and  $\eta$ .

The duality of the filtrations follows by looking at the filtrations on  $A^n$  and  $C^n$ .

#### Corollary (2.3.7)

$\mathcal{A} = \text{Hom}(SC, R)$  as filtered  $R$ -modules.

Proof: Use Theorem I of [41].

#### Corollary (2.3.8)

For  $n \geq p$ ,

$$(SC)_p = (S^n C)_p = \text{Hom}(S^n A / (S^n A)_{p+1}, R) = \text{Hom}(SA / (SA)_{p+1}, R).$$

Proof: Follows from (2.3.4), 2.3.6), and (2.2.7).

#### Lemma (2.3.9)

Let  $N$  be an increasingly filtered free  $R$ -module

$$N_0 \subset N_1 \subset \dots \subset N, \text{ with } N_p \text{ a free finitely}$$

generated direct summand of  $N$ , for each  $p$ , and  $N = \varinjlim_p \{N_p\}$ .

Let  $M = \text{Hom}(N, R)$  decreasingly filtered by  $M_p = \text{Annih}(N_{p-1})$

then  $\text{Hom}(N \otimes N, R) = M \hat{\otimes} M$ .

Proof: By V.4.1 of [36], it follows that

$$M/M_p = \text{Hom}(N_{p-1}, R),$$

and since tensor product commutes with direct limits,

$$\begin{aligned} [49] \quad 5.1.9, \quad \text{Hom}(N \otimes N, R) &= \text{Hom}(\varinjlim_{p,q} \{N_p \otimes N_q\}, R), \\ &= \varinjlim_{p,q} \{\text{Hom}(N_p, R) \otimes \text{Hom}(N_q, R)\}, \\ &= M \hat{\otimes} M, \text{ see (2.1.3)}. \end{aligned}$$

Corollary (2.3.10)

If  $K$  is an increasingly filtered Hopf algebra, with filtration satisfying the conditions of (2.3.9), then  $H = \text{Hom}(K, R)$  is a completed filtered Hopf algebra.

Proof: Immediate.

As an application of this corollary, note that if  $X$  is an  $H$ -space such that  $H_*(X)$  is torsion free of finite type then  $K_*(X)$  is a Hopf algebra over the integers, and the skeleton filtration,  $K_*(X)_p = \text{Im}\{K_*(X^p) \longrightarrow K_*(X)\}$ , satisfies the conditions of (2.3.9) since  $K_*(\ )$  is additive, see [38]. Then  $K^*(X) = \underline{K}^*(X) = \text{Hom}(K_*(X), \mathbb{Z})$ , by [17] and [28] (2.3), is the dual completed Hopf algebra.



Proposition (2.3.11)

If  $C$  is a filtered coalgebra and  $A$  the dual filtered algebra, then  $\mathcal{S}A$  is the dual completed Hopf algebra to  $SC$ .

If  $K, H$  are as in (2.3.10) and  $f: H \rightarrow A$  is dual to  $g: C \rightarrow K$ , then  $\mathcal{S}f$  of (2.2.20) is dual to  $Sg$  of (2.3.5).

Proof: Follows from (2.3.4), (2.3.7), (2.3.9), and the definitions.

Note that  $(SC)_p$  is not a subalgebra of  $SC$ , nor  $\mathcal{S}A/(\mathcal{S}A)_{p+1}$  a quotient coalgebra of  $\mathcal{S}A$ , so

$$(SC)_p = \text{Hom}(\mathcal{S}A/(\mathcal{S}A)_{p+1}, R),$$

as in (2.3.8), as coalgebras only. But products which do lie in  $(SC)_p$  may be computed by duality with the diagonal in  $\mathcal{S}A$ , modulo  $(\mathcal{S}A)_{p+1}$ .

§2.4 A Generating Map for K-theory

In [14], Bott has introduced for each compact Lie group  $G$  a generating variety  $V$ , and a map  $g: V \rightarrow \Omega G$ , which has the property that  $\text{Im}\{g_*: H_*(V) \rightarrow H_*(\Omega G)\}$  generates  $H_*(\Omega G)$  multiplicatively. We show how this extends to K-theory. We also generalise a theorem of Bott which enables one to compute  $H_*(\Omega G)$  from  $H^*(V)$  to K-theory.

Theorem (2.4.1)

If  $g: V \rightarrow \Omega G$  is the generating map of Bott [14], then  $\text{Im}\{g_*: K_*(V) \rightarrow K_*(\Omega G)\}$  generates  $K_*(\Omega G)$  multiplicatively.

Proof: Let  $\{^1E_{p,q}^r\}$  and  $\{^2E_{p,q}^r\}$  be the  $K_*$ -theory Atiyah-Hirzebruch spectral sequences for the spaces  $V$  and  $\Omega G$ , respectively. By naturality the second is a multiplicative spectral sequence.

Since the homology of both  $V$  and  $\Omega G$  occurs only in even dimensions, [13], both spectral sequences collapse. Since we are considering  $K_*$ -theory there are no convergence problems, see [24] p.146.

The map  $g: V \rightarrow \Omega G$  induces a map  $\{g_{p,q}^r: ^1E_{p,q}^r \rightarrow ^2E_{p,q}^r\}$  of the spectral sequences.

On the  $E^2$ -terms it is just the map

$$g_{*,0}^2 = g_*: H_*(V) \rightarrow H_*(\Omega G).$$

So, by Bott's result and the collapsing of the spectral sequences,  $\text{Im}\{g_{*,*}^\infty: ^1E_{*,*}^\infty \rightarrow ^2E_{*,*}^\infty\}$  generates  $^2E_{*,*}^\infty$  multiplicatively.

Let  $A \subset K_*(\Omega G)$  be the subring generated by the image of  $g_*: K_*(V) \rightarrow K_*(\Omega G)$ , we show, by induction on  $p$ , that  $K_*(\Omega G)_p \subset A$ . (Recall if  $X$  is a CW-complex  $K_*(X)_p = \text{Im}\{K_*(X^p) \rightarrow K_*(X)\}$ .)

Since  $H_*(\Omega G)$  is even-dimensional, the spectral sequence tells us that  $K_*(\Omega G)_{2p} = K_*(\Omega G)_{2p+1}$  so we need only consider the even filtration terms. Trivially  $K_*(\Omega G)_0 \subset A$ , assume now that  $K_*(\Omega G)_{2p-2} \subset A$ .

Let  $x \in K_*(\Omega G)_{2p}$ , denote by  $\bar{x}$  its class in  $K_*(\Omega G)_{2p}/K_*(\Omega G)_{2p-2} \cong {}^2E_{2p,*}^\infty$ . We can write

$$\bar{x} = \sum_i (g_* a_i^1) \dots (g_* a_i^{n_i}),$$

for some  $a_i^j \in {}^1E_{2p_i^j,*}^\infty \cong K_*(V)_{2p_i^j}/K_*(V)_{2p_i^j-2}$ , where for each  $i$   $\sum_{j=1}^{n_i} p_i^j = p$ .

For each  $i, j$  pick some  $y_i^j \in K_*(V)_{p_i^j}$  such that  $\overline{y_i^j} = a_i^j$ , then  $g_* a_i^j = \overline{g_* y_i^j}$  and

$$\bar{x} = \overline{\sum_i (g_* y_i^1) \dots (g_* y_i^{n_i})}.$$

So  $x - \sum_i (g_* y_i^1) \dots (g_* y_i^{n_i}) \in K_*(\Omega G)_{2p-2} \subset A$ , by induction,

but  $\sum_i (g_* y_i^1) \dots (g_* y_i^{n_i}) \in A$ , by definition, thus  $x \in A$ .

This completes the induction.

Since  $K_*( )$  is additive, by [38],  $K_*(\Omega G) = \bigcup_p K_*(\Omega G)_p$

and the theorem follows.

#### Corollary (2.4.2)

The map  $g^*: K^*(\Omega G) \longrightarrow K^*(V)$  maps the primitive module of  $K^*(\Omega G)$  into  $K^*(V)$  as a direct summand.

**Proof:** By [28] (2.3), using always [17] to identify  $\underline{K}^*(\Omega G)$  and  $K^*(\Omega G)$ , we have  $K^*(\Omega G) = \text{Hom}(K_*(\Omega G), \mathbb{Z})$ ,  $K^*(V) = \text{Hom}(K_*(V), \mathbb{Z})$ .

The theorem implies that the composition

$$\tilde{K}_*(V) \xrightarrow{\mathcal{E}^*} \tilde{K}_*(\Omega G) \longrightarrow QK_*(\Omega G)$$

is onto, where  $QK_*(\Omega G)$  is the indecomposable module. Thus

$$\tilde{K}_*(V) \xrightarrow{\mathcal{E}^*} \tilde{K}_*(\Omega G) \longrightarrow QK_*(\Omega G) \longrightarrow \frac{QK_*(\Omega G)}{\text{Torsion}} \quad (2.4.3)$$

is onto and split, since  $QK_*(\Omega G)/\text{Torsion}$  is free. The dual to (2.4.3) is

$$\begin{array}{ccccccc} \tilde{K}^*(V) & \xleftarrow{\mathcal{E}^*} & \tilde{K}^*(\Omega G) & \xleftarrow{\quad} & PK^*(\Omega G) & \xleftarrow{\quad} & \text{Hom}\left(\frac{QK_*(\Omega G)}{\text{Torsion}}, \mathbb{Z}\right), \\ & & & & \parallel & & \\ & & & & \text{Hom}(QK_*(\Omega G), \mathbb{Z}) & & \end{array}$$

which is thus a split monomorphism, proving the corollary.

Suppose now that we apply the constructions of §2.2 and §2.3 to  $K^*(V)$  and  $K_*(V)$  respectively. We give  $K^*(V)$  and  $K_*(V)$  the skeleton filtrations and the standard augmentation and unit,  $V$  is a homogeneous space and so has a base point and  $g$  is a map of based spaces, [14].

We obtain the following two diagrams

$$(2.4.4) \quad \begin{array}{ccc} & SK^*(V) & \\ \nearrow S\mathcal{E}^* & \downarrow & \\ K^*(\Omega G) & \xrightarrow{\mathcal{E}^*} & K^*(V), \end{array}$$

$$(2.4.5) \quad \begin{array}{ccc} & SK_*(V) & \\ \nwarrow S\mathcal{E}_* & \uparrow & \\ K_*(\Omega G) & \xleftarrow{\mathcal{E}_*} & K_*(V), \end{array}$$

where  $\mathcal{S}g^*$  and  $Sg_*$  are dual filtered Hopf algebra homomorphisms, see (2.3.11).

Theorem (2.4.6)

In the diagram (2.4.4),  $\mathcal{S}g^*$  identifies  $K^*(\Omega G)$  with the rational closure of the subring of  $\mathcal{S}K^*(V)$  generated by  $\{1, i(g^*PK^*(\Omega G))\}$ , where  $i: \tilde{K}^*(V) \longrightarrow \mathcal{S}K^*(V)$  is the map of (2.2.15), and  $PK^*(\Omega G)$  is the primitive module.

Note: The rational closure of a subring is the smallest subring containing it which is additively a direct summand.

Proof: By Theorem (2.4.1) and the definition of  $Sg_*$  in (2.4.5), dual to (2.2.20),  $Sg_*: \mathcal{S}K_*(V) \longrightarrow K_*(\Omega G)$  is onto. Since  $K_*(\Omega G)$  is torsion-free by duality we get that

$$\mathcal{S}g_*: K^*(\Omega G) \longrightarrow \mathcal{S}K^*(V)$$

is the inclusion of a direct summand.

Since  $K^*(\Omega G) \otimes \mathbb{Q} \cong H^{**}(\Omega G; \mathbb{Q})$ , and  $H^*(\Omega G; \mathbb{Q})$  is a graded Hopf algebra, by [39],  $K^*(\Omega G) \otimes \mathbb{Q}$  is the closure, with respect to the filtration topology, of the subring generated by 1 and the primitive elements. Hence 1 and  $g^*(PK^*(\Omega G))$  generate rationally a dense subring of  $K^*(\Omega G)$ .

Noting that a direct summand of a completely filtered module is closed in the filtration topology, and that, by

**Corollary (2.2.22),**  $g^*(PK^*(\Omega G)) = i(g^*PK^*(\Omega G))$  the result follows.

**Corollary (2.4.7)**

$$g^* \text{ induces } \frac{K^*(\Omega G)}{K^*(\Omega G)_{n+1}} \longrightarrow \frac{\mathcal{S}K^*(V)}{(\mathcal{S}K^*(V))_{n+1}}$$

which identifies  $K^*(\Omega G)/K^*(\Omega G)_{n+1}$  with the rational closure of the subring of  $\mathcal{S}K^*(V)/(\mathcal{S}K^*(V))_{n+1}$  generated by  $\{1, i(g^*PK^*(\Omega G)) \text{ modulo } (\mathcal{S}K^*(V))_{n+1}\}$ .

**Proof:** By (2.2.20)  $\mathcal{S}g^*$  is a filtered map, and thus the required map is induced.

From the Atiyah-Hirzebruch spectral sequences and using (2.2.23), we see that additively

$$K^*(\Omega G)/K^*(\Omega G)_{n+1} = \sum_{i=0}^n H^i(\Omega G),$$

$$\text{and } \mathcal{S}K^*(V)/(\mathcal{S}K^*(V))_{n+1} = \sum_{i=0}^n (\mathcal{S}H^*(V))^i,$$

where  $(\mathcal{S}H^*(V))^i$  is the  $i$ -th dimensional component of the graded algebra  $\mathcal{S}H^*(V)$ . The map between these is just the cohomology map  $\mathcal{S}g^*$  up to dimension  $n$ . It is therefore the inclusion of a direct summand, by 6.1 of [14]. The corollary now follows from the theorem.

If we know  $K^*(V)$  and  $\text{Im}\{g^*|: PK^*(\Omega G) \longrightarrow \tilde{K}^*(V)\}$  we may proceed to compute  $K_*(\Omega G)$  as follows.

Let  $n$  be the dimension of  $V$ , then, by Theorem (2.4.1)

and the known results on  $K_*(\Omega G) \otimes \mathbb{Q} \cong H_*(\Omega G; \mathbb{Q})$ ,  $K_*(\Omega G)$  can have no relations in filtration greater than  $n$ , since  $K_*(V)_n = K_*(V)$ . It is therefore sufficient to calculate  $K_*(\Omega G)_n$ , or by duality  $K^*(\Omega G)/K^*(\Omega G)_{n+1}$ , then using (2.4.7) and (2.2.7) this is given as the rational closure of the subring generated by  $\{1, i_n(g^*PK^*(\Omega G))\}$  in

$$\mathcal{S}^n K^*(V) / (\mathcal{S}^n K^*(V))_{n+1}.$$

The diagonal modulo  $(\mathcal{S}^n K^*(V))_{n+1}$ , then enables us to compute the multiplicative relations in  $K_*(\Omega G)$ .

We will apply this procedure in §2.6.

## §2.5 Adams Operations

For a CW-complex  $X$ , we have the operations

$$\psi^k: K^0(X) \longrightarrow K^0(X) \quad k \in \mathbb{Z},$$

of [1]. The  $\psi^k$  also act on  $K_0(X)$ , and for  $X$  such that  $K^0(X)$  is torsion-free the action is dual to that on  $K^0(X)$ . We may write this action also on the left since the  $\psi^k$  commute with each other.

Note: We write  $K^0$ ,  $K_0$  rather than  $K_*$ ,  $K^*$  when dealing with the Adams operations because of their instability, see [1] Corollary 5.3.

Lemma (2.5.1)

If  $X, Y$  are CW-complexes,  $\alpha \in K(X)$ ,  $\beta \in K(Y)$ , then  
 $\psi^k \alpha \times \psi^k \beta = \psi^k(\alpha \times \beta)$  in  $K(X \times Y)$ . ( $K = K^0$  or  $K_0$  throughout.)

Proof: Follows immediately from [1] theorem 5.1, (ii).

Proposition (2.5.2)

Let  $X$  be a CW-complex,

1)  $\psi^k: K^0(X) \rightarrow K^0(X)$  is a ring homomorphism.

2) If  $K_0(X)$  is torsion-free,  $\psi^k: K_0(X) \rightarrow K_0(X)$  is a coalgebra homomorphism.

3) If  $X$  is an H-space and  $K^0(X)$  is torsion-free, then  $\psi^k: K^0(X) \rightarrow K^0(X)$  and  $\psi^k: K_0(X) \rightarrow K_0(X)$  are Hopf algebra homomorphisms.

Proof: Follows from the lemma by naturality.

Let  $A = K^0(V)$  as algebra over  $\mathbb{Z}$ , and define an action of  $\psi^k$  on  $A^n$ , by  $\psi^k(a_1 \otimes \dots \otimes a_n) = (\psi^k a_1) \otimes \dots \otimes (\psi^k a_n)$ ,  $a_i \in A$ .

Proposition (2.5.3)

The action of the  $\psi^k$  on  $A^n$  restricts to  $\zeta^n A$  and defines in the limit an action on  $\zeta A$  such that  $i: \tilde{A} \rightarrow \tilde{\zeta} A$  commutes with the  $\psi^k$ .

If  $g: V \rightarrow X$  is a continuous map and  $X$  an H-space such that  $K^0(X)$  is torsion free with commutative diagonal,



then the induced map  $\mathcal{S}g^*:K^0(X) \longrightarrow \mathcal{S}K^0(V)$  of (2.2.20) commutes with the action of the  $\psi^k$ .

Proof: Clearly  $\psi^k$  maps symmetric elements to symmetric elements in  $A^n$ . If  $a_1 \otimes \dots \otimes a_n \in A^n$ ,

$$\varepsilon^n \cdot \psi^k(a_1 \otimes \dots \otimes a_n) = \varepsilon(\psi^k a_n) \psi^k(a_1 \otimes \dots \otimes a_{n-1}),$$

but  $\varepsilon(\psi^k a_n) = \varepsilon(a_n)$ , thus the first statement is proved.

To show that  $i$  and  $\psi^k$  commute, it is sufficient to prove that  $i_n: \tilde{A} \longrightarrow \hat{\mathcal{S}}^n A$  and  $\psi^k$  do. But, if  $a \in \tilde{A}$ ,

$$\begin{aligned} \psi^k \cdot i_n(a) &= \psi^k(a \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes a), \\ &= \psi^k a \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes \psi^k a, \\ &= i_n(\psi^k a). \end{aligned}$$

$\mathcal{S}g^*:K^0(X) \longrightarrow \mathcal{S}K^0(V)$  is defined as the limit of  $K^0(X) \xrightarrow{\Delta_n} \hat{\mathcal{S}}^n K^0(X) \xrightarrow{\hat{\mathcal{S}}^n g^*} \mathcal{S}^n K^0(V)$ , see (2.2.20). By induction on Proposition (2.5.2) 3) the  $\psi^k$  commute with  $\Delta_n$ , and by naturality with  $\hat{\mathcal{S}}^n g^*$ . Hence the result.

Thus the action of the Adams operations on  $K^0(\Omega G)$  may be computed from that on  $K^0(V)$  using (2.5.3) and (2.4.6). The action on  $K_0(\Omega G)$  can be obtained by duality.

#### Application (2.5.4)

As Bott remarks in [14], the space  $\Omega S^{2n+1}$  is a homology commutative H-space, thus  $K^*(\Omega S^{2n+1})$  is a

commutative completed Hopf algebra, and the map

$$g: S^{2n} \longrightarrow \Omega S^{2n+1},$$

given as the adjoint of the identity map, is a generating map for homology, and hence for K-theory, same proof as (2.4.1). Let  $\alpha$  be the generator of  $\tilde{K}^*(S^{2n})$ , then we have the following result, using [1] Corollary 5.2,

$$K^*(\Omega S^{2n+1}) \cong \mathbb{S}K^*(S^{2n})$$

is a divided power series algebra on one primitive generator

$\beta = i(\alpha)$ , and the action of the Adams operations is

determined by  $\psi^k \beta = k^n \beta$ .

## §2.6 Applications

We apply the results of §2.4 to calculate  $K_*(\Omega G)$ , for  $G = SU(n+1)$ , and the case  $G = Sp(2)$ . We have not been able to extend the application of our results further.

### Theorem (2.6.1)

As Hopf algebra

$$K_*(\Omega SU(n+1)) = \mathbb{Z}[\sigma_1, \dots, \sigma_n], \quad \sigma_i \in K_0,$$

with diagonal

$$\Delta \sigma_k = \sum_{i+j=k} \sigma_i \otimes \sigma_j, \quad \text{putting } \sigma_0 = 1.$$

Proof: The generating variety for  $SU(n+1)$  is complex

projective  $n$ -space  $\mathbb{CP}^n$ , [14] 5, and

$$K^*(\mathbb{CP}^n) \cong \frac{\mathbb{Z}[\alpha]}{(\alpha^{n+1})}, \quad \alpha \in K^0, \quad [1] \text{ 7.2.}$$

From the rational result we know that  $PK^*(\Omega SU(n+1))$  is a free  $\mathbb{Z}$ -module on  $n$  generators. Since, by (2.4.2),  $g^*|: PK^*(\Omega SU(n+1)) \longrightarrow \tilde{K}^*(\mathbb{CP}^n)$  is the inclusion of a direct summand,  $g^*|: PK^*(\Omega SU(n+1))$  must be an isomorphism. Thus the result of Theorem (2.4.4) applied to this case is that  $K^*(\Omega SU(n+1))$  is isomorphic to the rational closure of the subalgebra of  $\mathbb{S}K^*(\mathbb{CP}^n)$  generated by  $i(\tilde{K}^*(\mathbb{CP}^n))$ , but this is clearly the whole algebra. So

$$K^*(\Omega SU(n+1)) \cong \mathbb{S}K^*(\mathbb{CP}^n).$$

The dual result is that  $K_*(\Omega SU(n+1)) \cong SK_*(\mathbb{CP}^n)$ , which is easily seen to be a polynomial algebra on  $\tilde{K}_*(\mathbb{CP}^n)$ . If we take  $\sigma_i \in \tilde{K}_*(\mathbb{CP}^n)$  dual to  $\alpha^i \in \tilde{K}^*(\mathbb{CP}^n)$ , we find that

$$\langle \alpha^i \otimes \alpha^j, \Delta \sigma_k \rangle = \langle \alpha^{i+j}, \sigma_k \rangle = \delta_{i+j,k},$$

which gives the diagonal.

We may compute the Adams operations in  $K_0(\Omega SU(n+1))$  from the result of [1] 7.2 on  $K^0(\mathbb{CP}^n)$ . We have

$$\psi^k \alpha^i = ((1+\alpha)^k - 1)^i,$$

so by duality  $\psi^k \sigma_1 = k \sigma_1$ ,  $\psi^k \sigma_2 = k^2 \sigma_2 + \frac{k(k-1)}{2} \sigma_1$ , etc., and these hold in  $K_0(\Omega SU(n+1))$  by §2.5.

We turn now to  $\Omega\mathrm{Sp}(2)$ , its generating variety is the space  $\mathrm{Sp}(2)/\mathrm{U}(2)$ , see [14] 5.

Lemma (2.6.2)

$$K^*\left(\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}\right) \cong \frac{\mathbb{Z}[a,c]}{(c^2, a^2-2c+ac)}, \quad a, c \in K^0,$$

$$H^*\left(\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}; \mathbb{Q}\right) \cong \frac{\mathbb{Q}[u]}{(u^4)} \quad \dim u = 2,$$

and where  $u$  is chosen to be an integral generator

$$\begin{aligned} \mathrm{cha} &= u - \frac{u^3}{12}, \\ \mathrm{ch} c &= \frac{u^2}{2} - \frac{u^3}{4}. \end{aligned}$$

Proof: We consider the composition

$$R(\mathrm{U}(2)) \xrightarrow{\alpha} K^*\left(\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}\right) \xrightarrow{\mathrm{ch}} H^*\left(\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}; \mathbb{Q}\right),$$

where  $\alpha$  is the epimorphism of [8] 5.7. Since  $H^*\left(\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}\right)$  is torsion-free, [13],  $\mathrm{ch}$  is a monomorphism, and so  $K^*\left(\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}\right)$  is identified with the image of the above composition.

By [12] 10.3, this is equal to the composition

$$R(\mathrm{U}(2)) \xrightarrow{\chi} H^{**}(B_T; \mathbb{Q})^{W(\mathrm{U}(2))} = H^{**}(B_{\mathrm{U}(2)}; \mathbb{Q}) \xrightarrow{\rho} H^*\left(\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}; \mathbb{Q}\right)$$

where we write  $W(G)$  for the Weyl group of  $G$ ,

$T \subset \mathrm{U}(2) \subset \mathrm{Sp}(2)$  is a maximal torus, and  $\chi$  is the character

homomorphism.  $\rho$  is induced from a map  $\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)} \longrightarrow B_{\mathrm{U}(2)}$ ,

but, under the Borel description, [9], it is also the

natural projection

$$H^{**}(B_T; \mathbb{Q})^{W(U(2))} \longrightarrow \frac{H^{**}(B_T; \mathbb{Q})^{W(U(2))}}{I_{Sp(2)}} = H^*\left(\frac{Sp(2)}{U(2)}; \mathbb{Q}\right),$$

where  $I_{Sp(2)}$  is the ideal generated by elements of positive dimension in  $H^{**}(B_{Sp(2)}; \mathbb{Q}) = H^{**}(B_T; \mathbb{Q})^{W(Sp(2))}$ .

Choosing suitable generators we have

$$H^{**}(B_T; \mathbb{Q}) = \mathbb{Q}[[x_1, x_2]] \quad \dim x_i = 2, \quad i=1, 2.$$

$$H^{**}(B_T; \mathbb{Q})^{W(U(2))} = \mathbb{Q}[[\sigma_1, \sigma_2]] \quad \sigma_1 = x_1 + x_2, \quad \sigma_2 = x_1 x_2.$$

$$H^{**}(B_T; \mathbb{Q})^{W(Sp(2))} = \mathbb{Q}[[\tau_1, \tau_2]] \quad \tau_1 = x_1^2 + x_2^2, \quad \tau_2 = (x_1 x_2)^2.$$

$$\begin{aligned} \text{Thus } H^*\left(\frac{Sp(2)}{U(2)}; \mathbb{Q}\right) &= \frac{\mathbb{Q}[\sigma_1, \sigma_2]}{(\tau_1, \tau_2)} \\ &= \frac{\mathbb{Q}[u]}{(u^4)}, \quad \text{putting } u = \sigma_1 = x_1 + x_2. \end{aligned}$$

$$\begin{aligned} \text{And } \rho: \sigma_1 &\longmapsto u, \\ &: \sigma_2 \longmapsto \frac{u^2}{2}. \end{aligned}$$

Now, by [2] 7.3,

$$R(U(2)) = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_2^{-1}],$$

$$\text{and } \chi(\lambda_1) = \exp(x_1) + \exp(x_2),$$

$$\chi(\lambda_2) = \exp(x_1 + x_2).$$

Therefore

$$\begin{aligned} \text{ch}_*\alpha(\lambda_1) &= \rho_*\chi(\lambda_1) = \rho(2 + (x_1 + x_2) + \frac{(x_1^2 + x_2^2)}{2!} + \frac{(x_1^3 + x_2^3)}{3!} + \dots) \\ &= 2 + u - \frac{u^3}{12}, \end{aligned}$$

$$\text{and } \text{ch}_*\alpha(\lambda_2) = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} = \exp(u).$$

$$\begin{aligned} \text{Then } \text{ch}_*\alpha(\lambda_2^{-1}) &= \exp(-u) = (1 + (\exp u - 1))^{-1} \\ &= 1 - (\exp u - 1) + (\exp u - 1)^2 - (\exp u - 1)^3, \end{aligned}$$

since  $(\exp u - 1)^n = 0$ , for  $n \geq 4$ . Thus  $\text{ch.}\alpha(\lambda_2^{-1})$  is contained in the subring generated by  $\text{ch.}\alpha(\lambda_2)$ . Hence  $K^*(\frac{\text{Sp}(2)}{\text{U}(2)})$  is identified with the subring of  $\mathbb{Q}[u] / (u^4)$  generated by  $1, u - \frac{u^3}{12}, u + \frac{u^2}{2} + \frac{u^3}{6}$ . Letting

$$\begin{aligned} a &= u - \frac{u^3}{12}, \\ \text{and } c &= (u + \frac{u^2}{2} + \frac{u^3}{6}) - (u - \frac{u^3}{12}) \\ &= \frac{u^2}{2} + \frac{u^3}{4}, \end{aligned}$$

it is easy to verify that  $a^2 = 2c - ac$  and  $c^2 = 0$ , then  $\{1, a, c, ac\}$  provides a basis for  $K^*(\frac{\text{Sp}(2)}{\text{U}(2)})$  showing that there are no more relations. This proves the lemma.

#### Corollary (2.6.3)

The skeleton filtration of  $A = K^*(\frac{\text{Sp}(2)}{\text{U}(2)})$  has

$$A_{2q-1} = A_{2q} \text{ for all } q \text{ and}$$

$$A_2 = (a, c),$$

$$A_4 = (c),$$

$$A_6 = (ac),$$

$$A_8 = 0.$$

Proof: Simple application of [8] 2.5.

#### Lemma (2.6.4)

The Adams operations on  $K^0(\frac{\text{Sp}(2)}{\text{U}(2)})$  are given by

$$\psi^k a = ka - \frac{k(k-1)(k+1)}{6} ac,$$

$$\psi^k c = k^2 c + 2k^2(k-1) ac.$$

Proof: By Theorem 5.1 (vi) of [1], if  $\alpha \in K^0(X)$

$$\text{ch}^q(\psi^k \alpha) = k^q \text{ch}^q(\alpha),$$

where  $\text{ch}^q$  is the  $2q$ -dimensional component of the Chern character. So if we write

$$\psi^k a = A_k a + B_k c + C_k ac$$

then (2.6.2) shows that

$$\text{ch}(\psi^k a) = A_k u + \frac{B_k}{2} u^2 + \left( \frac{B_k}{4} - \frac{A_k}{12} + \frac{C_k}{2} \right) u^3.$$

So we get  $A_k = k$ ,  $\frac{B_k}{2} = 0$ ,  $\frac{C_k}{2} - \frac{k}{12} = \frac{-k^3}{12}$ , which yields the result for  $\psi^k a$ , similarly for  $\psi^k c$ .

Theorem (2.6.5)

$$K_*(\Omega \text{Sp}(2)) \cong \mathbb{Z}[u, v], \quad u, v \in K_0,$$

with diagonal  $u = u \otimes 1 + 1 \otimes u$ ,

$$v = v \otimes 1 + u^2 \otimes u - u \otimes u + u \otimes u^2 + 1 \otimes v.$$

Proof: We must first find the image of

$$g^*|: PK^*(\Omega \text{Sp}(2)) \longrightarrow \tilde{K}^*\left(\frac{\text{Sp}(2)}{U(2)}\right).$$

Since  $H^*(\Omega \text{Sp}(2); \mathbb{Q})$  has primitive generators in dimensions 2 and 6, the image of

$$g^*|: PH^*(\Omega \text{Sp}(2); \mathbb{Q}) \longrightarrow \tilde{H}^*\left(\frac{\text{Sp}(2)}{U(2)}; \mathbb{Q}\right)$$

is the submodule generated by  $\{u, u^3\}$ . Thus by Lemma (2.6.2) and (2.4.2) the required image in  $\tilde{K}^*\left(\frac{\text{Sp}(2)}{U(2)}\right)$  is the submodule generated by  $\{a, ac\}$ .

By Proposition (2.4.7),  $K^*(\Omega Sp(2))/K^*(\Omega Sp(2))_7$  is identified with the rational closure of the subalgebra of

$$\mathcal{S}^n_{K^*}(\frac{Sp(2)}{U(2)})/(\mathcal{S}^n_{K^*}(\frac{Sp(2)}{U(2)}))_7$$

generated by  $\{1, i(a), i(ac)\}$ , for  $n \geq 6$ .

We abuse the notation of §2.2 and write  $[x \otimes y]$  for  $[x \otimes y \otimes 1 \otimes \dots \otimes 1]$  etc. Recall that the brackets  $[ ]$  indicate summation over distinct permutations. Put

$$\eta = i(a) = [a], \zeta = i(ac) = [ac].$$

$\eta$  is in filtration 2,  $\zeta$  in filtration 6, thus, modulo  $(\mathcal{S}^n_{K^*}(\frac{Sp(2)}{U(2)}))_7$ , we have

$$\begin{aligned} \eta^4 &= \eta\zeta = \zeta^2 = 0, \\ \eta^2 &= [a^2] + 2[a \otimes a], \\ &= 2[c] - [ac] + 2[a \otimes a], \\ \eta^3 &= [a^3] + 3[a^2 \otimes a] + 6[a \otimes a \otimes a], \\ &= 2[ac] + 6[c \otimes a] + 6[a \otimes a \otimes a]. \end{aligned}$$

Thus  $K^*(\Omega Sp(2))/K^*(\Omega Sp(2))_7$  has basis

$$\{1, \eta, \frac{\eta^2 + \zeta}{2}, \frac{\eta^3 - 2\zeta}{6}, \zeta\}.$$

In the dual  $K_*(\Omega Sp(2))_6$  take  $\{1, u\}$  as the beginning of a dual basis, then

$$\langle \eta^2, u^2 \rangle = \langle \Delta \eta^2, u \otimes u \rangle = 2, \quad \eta = i(a) \text{ is primitive,}$$

$$\langle \eta, u^2 \rangle = \langle \zeta, u^2 \rangle = \langle \eta^3, u^2 \rangle = 0.$$

$$\langle \eta^3, u^3 \rangle = 6,$$



$$\langle \eta, u^3 \rangle = \langle \zeta, u^3 \rangle = \langle \eta^2, u^3 \rangle = 0.$$

Choose  $v$  such that

$$\langle \zeta, v \rangle = 1,$$

$$\langle \eta, v \rangle = \langle \frac{\eta^2 + \zeta}{2}, v \rangle = \langle \frac{\eta^3 - 2\zeta}{6}, v \rangle = 0,$$

then  $\{1, u, u^2, u^3, v\}$  is a basis for  $K_*(\Omega\mathrm{Sp}(2))_6$ . The multiplicative structure of  $K_*(\Omega\mathrm{Sp}(2))$  follows by the remarks at the end of §2.4.

For the diagonal,  $u$  is clearly primitive since it is in filtration 2, but  $\langle \eta^2, v \rangle = -1$ ,  $\langle \eta^3, v \rangle = 2$ , so  $\langle \eta \otimes \eta, \Delta v \rangle = -1 = \text{coefficient of } u \otimes u \text{ in } \Delta v$ ,

$$\langle \eta \otimes \frac{\eta^2 + \zeta}{2}, \Delta v \rangle = 1 = \text{coefficient of } u \otimes u^2 \text{ in } \Delta v,$$

and the theorem is proved.

Note: We obtain this result by a different method below, see (3.6.7).

Proposition (2.6.6)

On  $K_0(\Omega\mathrm{Sp}(2))$  the Adams operations are given by

$$\psi^k u = ku,$$

$$\psi^k v = k^3 v + \frac{k^2(k-1)}{2} u^2 - \frac{k(k-1)(k+1)}{6} u.$$

Proof: Use Lemma (2.6.4) and Proposition (2.5.3).

## §2.7 $K^*(\Omega G)$ via Representation Theory

We show in this section how we can obtain a description for  $K^*(\Omega G)$  in terms of the representation theory of  $G$ .

We suppose throughout that  $s:S^1 \longrightarrow G$  is a homomorphism, that  $T \subset G$  is a maximal torus of  $G$  containing the image of  $s$ , and that  $H$  is the centraliser of the image of  $s$  in  $G$ . Thus  $T \subset H \subset G$ .

We define the map

$$f_s: G/H \longrightarrow \Omega G,$$

$$\text{by } f_s(gH): t \longmapsto gs(t)g^{-1}, \text{ for } t \in S^1.$$

Our interest in this situation is the result of Bott, [14], that for certain maps  $s:S^1 \longrightarrow G$  called generating circles, the map  $f_s$  is the generating map of §2.4.

### Definition (2.7.1)

If  $\rho:T \longrightarrow U(1) = S^1$  is a one-dimensional complex representation of  $T$ , let  $d_s(\rho) = \text{degree}\{S^1 \xrightarrow{s} T \xrightarrow{\rho} S^1\} \in \mathbb{Z}$ , and let  $\hat{\theta}_s(\rho) = d_s(\rho)\rho \in R(T)$ . Since  $R(T)$  is additively generated by one-dimensional representations,  $\hat{\theta}_s$  extends linearly to  $\tilde{\theta}_s: R(T) \longrightarrow R(T)$ .

### Proposition (2.7.2)

$\tilde{\theta}_s: R(T) \longrightarrow R(T)$  is a derivation.

Proof: Let  $\rho_1, \rho_2$  be two one-dimensional representations of  $T$ , then  $d_s(\rho_1 \rho_2) = d_s(\rho_1) + d_s(\rho_2)$ , the result follows.

Definition (2.7.3)

If  $v \in H^2(B_T)$ , let  $\theta_s(v) \in \mathbb{Z}$ , be defined as follows.  
 $s: S^1 \longrightarrow T = \Omega B_T$  has adjoint  $\hat{s}: S^2 \longrightarrow B_T$ ; representing  $v$  as a map  $B_T \longrightarrow K(\mathbb{Z}, 2)$ , the composition

$$S^2 \longrightarrow B_T \longrightarrow K(\mathbb{Z}, 2)$$

determines an element of  $\pi_2(K(\mathbb{Z}, 2))$ , this is  $\theta_s(v)$ .

Since  $H^*(B_T)$  is multiplicatively generated by  $H^2(B_T)$ ,

$\theta_s$  extends uniquely to a derivation  $\theta_s: H^*(B_T) \longrightarrow H^*(B_T)$ ;

we can also consider  $\theta_s$  as a derivation of  $H^*(B_T; \mathbb{Q})$  or of  $H^{**}(B_T; \mathbb{Q})$ .

Proposition (2.7.4)

Let  $ch_\bullet \alpha$  be the composition

$$R(T) \xrightarrow{\alpha} K^*(B_T) \xrightarrow{ch} H^{**}(B_T; \mathbb{Q}), \quad \text{as in [8],}$$

then the following diagram commutes

$$\begin{array}{ccc} R(T) & \xrightarrow{ch_\bullet \alpha} & H^{**}(B_T; \mathbb{Q}) \\ \hat{\theta}_s \downarrow & & \downarrow \theta_s \\ R(T) & \xrightarrow{ch_\bullet \alpha} & H^{**}(B_T; \mathbb{Q}). \end{array}$$

Proof: Let  $\rho$  be a one-dimensional representation of  $T$ ,

then  $ch_\bullet \alpha(\rho) = \exp(x) = 1 + x + \frac{x^2}{2!} + \dots$ ,

where  $x \in H^2(B_T)$  is the first Chern class of  $\rho$ , determined in the following manner. The homomorphism  $\rho$  induces  $B_\rho: B_T \longrightarrow B_{U(1)} = K(\mathbb{Z}, 2)$ , then  $x$  is the element determined by the homotopy class of  $B_\rho$ .

Now  $\theta_s(x^i) = i\theta_s(x)x^{i-1}$ , since  $\theta_s$  is a derivation. Hence  $\theta_s(\exp(x)) = \theta_s(x)\exp(x)$ .

On the other hand it is clear that  $\theta_s(x) = d_s(\rho)$ , since the composition  $S^2 \xrightarrow{\hat{s}} B_T \xrightarrow{B_\rho} K(\mathbb{Z}, 2)$  can also be considered as the adjoint of the composition  $S^1 \xrightarrow{s} T \xrightarrow{\rho} S^1 = K(\mathbb{Z}, 1)$ .

Proposition (2.7.5)

$\hat{\theta}_s$  restricts to  $\tilde{\theta}_s: R(H) \longrightarrow R(H)$ .

Proof: Since  $R(H) = R(T)^{W(H)}$ , where  $W(H)$  is the Weyl group of  $H$  with respect to  $T$ , it is clearly sufficient to show that  $\tilde{\theta}_s$  is  $W(H)$ -equivariant, and this on one-dimensional representations. Recall that  $W(H) = N_H(T)/T$ , where  $N_H(T)$  is the normaliser of  $T$  in  $H$ , which acts by conjugation on  $T$ , and this induces the action on  $R(T)$ . So if  $w \in W(H)$  is represented by  $n \in N_H(T) \subset H$ ,  $d_s(w\rho)$  is the degree of the map  $S^1 \longrightarrow S^1$  given by

$$t \longmapsto \rho(ns(t)n^{-1}),$$

but, since  $H$  is the centraliser of the image of  $s$ ,

$d_s(w\rho) = d_s(\rho)$ . Hence  $\tilde{\theta}_s(w\rho) = w\tilde{\theta}_s(\rho)$ , proving the result.

Since  $R(G) = R(T)^{W(G)} \subset R(H)$ , (2.7.5) allows us to consider  $\hat{\theta}_S$  as a map  $R(G) \longrightarrow R(H)$ .

A similar proof, see [14] 7, shows that  $\theta_S$  restricts to  $H^{**}(B_G; \mathbb{Q}) \longrightarrow H^{**}(B_H; \mathbb{Q})$ .

Theorem (2.7.6)

If  $\beta: R(G) \longrightarrow \tilde{K}^*(G)$  is the map of [31] I §4,

$\sigma: \tilde{K}^*(G) \longrightarrow \tilde{K}^*(\Omega G)$  is the cohomology suspension,

and  $\alpha: R(H) \longrightarrow K^*(G/H)$  is the homomorphism of [8] 5,

then the following diagram commutes up to sign

$$\begin{array}{ccccc}
 R(G) & \xrightarrow{\beta} & \tilde{K}^*(G) & \xrightarrow{\sigma} & \tilde{K}^*(\Omega G) \\
 \hat{\theta}_S \downarrow & & & & \downarrow f_S^* \\
 R(H) & \xrightarrow{\alpha} & K^*(G/H) & & 
 \end{array}$$

Proof: Consider the diagram

$$\begin{array}{ccccccc}
 & & H^{**}(B_G; \mathbb{Q}) & \xrightarrow{\sigma^2} & \tilde{H}^{**}(\Omega G; \mathbb{Q}) & & \\
 & \nearrow \text{ch}_* \alpha & \downarrow \theta_S & & \nearrow \text{ch} & & \\
 R(G) & \xrightarrow{\sigma_* \beta} & \tilde{K}^*(\Omega G) & & & \downarrow f_S^* & \\
 \hat{\theta}_S \downarrow & & \downarrow f_S^* & & & & \\
 & \nearrow \text{ch}_* \alpha & H^{**}(B_H; \mathbb{Q}) & \xrightarrow{\quad} & H^*(G/H; \mathbb{Q}) & & \\
 R(H) & \xrightarrow{\alpha} & K^*(G/H) & & & \nearrow \text{ch} & 
 \end{array}$$

where  $\sigma^2$  denotes double suspension.

We note that all the faces of the cube except the

front one are known to commute at least up to sign. For the top one use Proposition 4.1 of [31], for the left hand end (2.7.4), and the commutativity of the back face is the rational cohomology version of the theorem which is Proposition 7.2 of [14]. For the others use naturality of the Chern character. Then since  $\text{ch}: K^*(G/H) \longrightarrow H^*(G/H; \mathbb{Q})$  is a monomorphism,  $H^*(G/H)$  being torsion-free, [13], a simple diagram chase shows that the front face commutes up to sign, proving the theorem.

Now suppose  $G$  is 1-connected, then the homotopy sequence of the fibration  $H \longrightarrow G \longrightarrow G/H$  shows that  $\pi_1(H) = \pi_2(G/H) = H_2(G/H)$  which is torsion free, [13]. Then  $H \subset G$  satisfies the conditions of Theorem 4.2 of [48], thus  $\alpha: R(H) \longrightarrow K^*(G/H)$  is the projection

$$R(H) \longrightarrow \frac{R(H)}{(I(G))} = R(H) \otimes_{R(G)} \mathbb{Z},$$

where  $(I(G))$  is the  $R(H)$ -ideal generated by the augmentation ideal of  $R(G)$ .

Proposition (2.7.7)

The primitive module  $PK^*(\Omega G) \subset \hat{K}^*(\Omega G)$  is the rational closure of the submodule  $\text{Im}\{\sigma.\beta: R(G) \longrightarrow \hat{K}^*(\Omega G)\}$ .

Proof: Follows immediately from the corresponding rational

cohomology result, [14] 7.1, and the fact that  $K^*(\Omega G)$  has no torsion.

Theorem (2.7.8)

If  $s$  is a generating circle, see [14],  $K^*(\Omega G)$  is identified with the rational closure of the subring of  $\mathcal{S}(R(H)/(I(G)))$  generated by 1 and the image of

$$R(G) \xrightarrow{\tilde{\theta}_s} R(H) \longrightarrow R(H)/(I(G)) \xrightarrow{i} \mathcal{S}(R(H)/(I(G))),$$

which is primitive.

Proof: Follows from (2.7.6), (2.7.7), and (2.4.6).

### Chapter 3

In this chapter we develop and extend a method due to T. Petrie, [40], for computing the complex bordism of the loop space of a Lie group. We then use the Conner-Floyd isomorphism to obtain the K-theory. The first five sections are concerned with setting up the general theory which is applied to specific examples in the last three.

#### §3.1 The bordism theory $\Omega_*^U(\ )$

We recall the definition of, and facts about, the generalised homology theory of complex bordism.

Let  $X$  be a space with the homotopy type of a CW-complex. If  $n \geq 0$ , we denote by  $\Omega_n^U(X)$  the set of bordism classes of maps  $f: M \rightarrow X$ , where  $M$  is an  $n$ -dimensional stably almost complex manifold.  $\Omega_n^U(X)$  is an abelian group under the operation of disjoint union, [22], [23]. Let  $\Omega_*^U(X)$  be the graded module. We write  $[M, f]$  for the bordism class of  $f: M \rightarrow X$ .

If  $X$  and  $Y$  are spaces, we have an exterior product

$$\Omega_n^U(X) \otimes \Omega_m^U(Y) \xrightarrow{\times} \Omega_{n+m}^U(X \times Y),$$

given by  $[M, f] \otimes [N, g] \mapsto [M \times N, f \times g]$ .



Taking  $X = Y = \text{point}$ , we see that  $\Omega_*^U(\text{point}) = \Omega_*^U$  is a graded ring over the ground ring  $\mathbb{Z}$ .

Taking  $X$  or  $Y = \text{point}$ ,  $\Omega_*^U(X)$  becomes a left and right graded module over the graded ring  $\Omega_*^U$ .

**Proposition (3.1.1)**

The exterior product factors to give a product

$$\Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) \xrightarrow{\times} \Omega_*^U(X \times Y).$$

**Proof:**  $\otimes_{\Omega_*^U}$  is defined by the exact sequence, see [39] 1,

$$\Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) \xrightarrow{S} \Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) \longrightarrow \Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) \longrightarrow 0,$$

where  $S: \alpha \otimes \varphi \otimes \beta \longmapsto \alpha \varphi \otimes \beta - \alpha \otimes \varphi \beta$ . It is trivial to check that  $S$  followed by the exterior product is the zero map.

**Proposition (3.1.2)**

The diagram

$$\begin{array}{ccc} \Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) & \xrightarrow{\times} & \Omega_*^U(X \times Y) \\ \downarrow T & & \downarrow t_* \\ \Omega_*^U(Y) \otimes_{\Omega_*^U} \Omega_*^U(X) & \xrightarrow{\times} & \Omega_*^U(Y \times X), \end{array}$$

commutes, where  $T: \alpha \otimes \beta \longmapsto (-1)^{\dim \alpha \cdot \dim \beta} \beta \otimes \alpha$ , and  $t: X \times Y \longrightarrow Y \times X$ ,  $(x, y) \longmapsto (y, x)$ .

**Proof:** Follows from the corresponding result on stably almost complex structures, [23] p.23.

**Corollary (3.1.3)**

If  $X$  is an H-space, then  $\Omega_*^U(X)$  has the structure

of a graded augmented algebra with unit over  $\Omega_*^U$ . If  $X$  is homotopy commutative, then  $\Omega_*^U(X)$  is graded commutative.

Proof: Define the product as the composition

$$\Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(X) \xrightarrow{x} \Omega_*^U(X \times X) \xrightarrow{m_*} \Omega_*^U(X),$$

where  $m: X \times X \longrightarrow X$  gives  $X$  its H-space structure. The augmentation  $\varepsilon: \Omega_*^U(X) \longrightarrow \Omega_*^U$  is induced from the trivial map  $X \longrightarrow \text{point}$ , and the unit  $\eta: \Omega_*^U \longrightarrow \Omega_*^U(X)$  from the inclusion of the identity of  $X$ .

The commuting of the appropriate structure diagrams, see [39], follows by naturality, and the statement about commutativity is a consequence of Proposition (3.1.2).

Suppose now that  $X$  is such that  $\Omega_*^U(X)$  is a projective  $\Omega_*^U$ -module of finite type, then the usual arguments, [33] or see [35], show that if  $Y$  is a CW-complex with a finite number of cells in each dimension the product

$$\Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) \xrightarrow{x} \Omega_*^U(X \times Y)$$

is an isomorphism.

Proposition (3.1.4)

Let  $X$  have the homotopy type of a CW-complex with finite skeletons and such that  $\Omega_*^U(X)$  is a projective  $\Omega_*^U$ -module of finite type, then  $\Omega_*^U(X)$  has a graded commutative coalgebra structure with unit over  $\Omega_*^U$ .

Proof: Define the diagonal as the composition

$$\Omega_*^U(X) \xrightarrow{\Delta_*} \Omega_*^U(X \times X) \xleftarrow[\cong]{\quad} \Omega_*^U(X) \otimes_{\Omega_*^U(X)} \Omega_*^U(X),$$

where  $\Delta: X \rightarrow X \times X$  is the diagonal map, and the unit by  $\Omega_*^U(X) \rightarrow \Omega_*$ , induced from  $X \rightarrow \text{point}$ .

It is trivial to verify that the appropriate diagrams commute. Commutativity is a corollary of Proposition (3.1.3).

Proposition (3.1.5)

Let  $X$  be an  $H$ -space satisfying the conditions of (3.1.4), then  $\Omega_*^U(X)$  has a graded Hopf algebra structure over  $\Omega_*^U$ .

Proof: All that remains to prove is that the diagonal is an algebra homomorphism, but just as in the proof of Proposition (2.1.10) this reduces to the trivial verification of a commutative diagram of spaces.

For example, if  $G$  is a 1-connected compact Lie group take  $X = \Omega G$ , the space of loops on  $G$ . Then, [13],  $H_*(X)$  is torsion-free and lies entirely in even dimensions, thus the bordism spectral sequence, [21],

$$H_p(X; \Omega_q^U) \rightarrow \Omega_*^U(X)$$

collapses and  $\Omega_*^U(X)$  is a free  $\Omega_*^U$ -module. Therefore  $\Omega_*^U(X)$

is a Hopf algebra; since  $X$  is homotopy commutative, the algebra is commutative.

A stably almost complex manifold  $M$  has canonical homology and  $K$ -theory orientation classes, which we denote by  $\alpha_M \in H_*(M)$ , and  $[M] \in K_*(M)$ , respectively. These define homomorphisms

$$\begin{aligned} \mu: \Omega_*^U(X) &\longrightarrow H_*(X), & [M, f] &\longmapsto f_*(\alpha_M) \\ \text{and } \mu: \Omega_*^U(X) &\longrightarrow K_*(X), & [M, f] &\longmapsto f_*([M]). \end{aligned}$$

The first is an epimorphism if  $H_*(X)$  is torsion-free, [24] (3.1), and the second induces

$$\tilde{\mu}: \Omega_*^U(X) \otimes_{\Omega_*^U} \mathbb{Z} \longrightarrow K_*(X),$$

which is an isomorphism for  $X$  a CW-complex with finite skeletons, [24] §9.

Proposition (3.1.6)

The composition

$$\Omega_*^U(X) \xrightarrow{\mu} K_*(X) \xrightarrow{\text{ch}} H_*(X; \mathbb{Q})$$

is given by  $[M, f] \longmapsto f_*(\text{todd}(M) \cap \alpha_M)$ .

Here  $\text{todd}(M)$  denotes the Todd polynomial, [29] 1.7, in the Chern classes of  $M$ , modified by a sign in dimensions  $2(2k+1)$ .

Proof: Let  $x \in K^*(X)$ , then, by the Lemma on p.119 of [50],

$$\langle f_*x, [M] \rangle = \langle \text{chf}^*x, \text{todd}(M), o_M \rangle.$$

So if  $\alpha = [M, f] \in \Omega_*^U(X)$  we have

$$\begin{aligned} \langle \text{ch}x, \text{ch}\mu\alpha \rangle &= \langle x, \mu\alpha \rangle, \\ &= \langle x, f_*[M] \rangle, \\ &= \langle \text{chf}^*x, \text{todd}(M), o_M \rangle, \\ &= \langle f^*\text{ch}x, \text{todd}(M) \cap o_M \rangle, [49], \text{Ch5, 6.18,} \\ &= \langle \text{ch}x, f_*(\text{todd}(M) \cap o_M) \rangle. \end{aligned}$$

But  $H_*(X; \mathbb{Q})$  and  $H^{**}(X; \mathbb{Q})$  are dual, and  $\text{ch}K^*(X)$  generates  $H^{**}(X; \mathbb{Q})$ , hence the result.

### §3.2 Characteristic Numbers

In this section we define characteristic numbers for bordism elements and examine their behaviour with respect to products.

If  $M$  is a stably almost complex manifold, we have the Chern classes  $c_i(M) \in H^{2i}(M)$ .

Let  $\omega = (i_1, \dots, i_k)$  be a partition of  $|\omega| = i_1 + \dots + i_k$ . If  $n \geq |\omega|$ , write  $s_\omega$  for the smallest symmetric polynomial in  $x_1, \dots, x_n$  containing the term  $x_1^{i_1} \dots x_k^{i_k}$ . Then if  $c_i = i$ -th elementary symmetric function of the  $x_1, \dots, x_n$  we can write  $s_\omega$  as a polynomial  $P_\omega(c_1, \dots, c_{|\omega|})$  in the  $c_1, \dots, c_{|\omega|}$ , which is independent of  $n$ .

Thus, using the cup-product in  $H^*(M)$ , we can define

$$s_\omega(M) = P_\omega(c_1(M), \dots, c_{|\omega|}(M)) \in H^{2|\omega|}(M).$$

We write  $s_0(M) = 1 \in H^0(M)$ .

If  $2|\omega| = \text{dimension of } M$ , we have

$$s_\omega[M] = s_\omega(M) \cap o_M = \langle s_\omega(M), o_M \rangle \in \mathbb{Z}.$$

Definition (3.2.1)

Let  $\alpha = [M, f] \in \Omega_p^U(X)$ , we define the class

$$S_\omega(\alpha) \in H_{p-2|\omega|}(X)$$

to be  $f_*(s_\omega(M) \cap o_M)$ .

For explicit calculations it is easier to deal with integers than homology elements, so we shall use

Definition (3.2.2)

Let  $\alpha = [M, f] \in \Omega_p^U(X)$ ,  $x \in H^q(X)$ , then define the characteristic number

$$S_\omega(\alpha)(x) = \langle f^*(x), s_\omega(M), o_M \rangle \in \mathbb{Z}.$$

Clearly  $S_\omega(\alpha)(x) = 0$  unless  $p = 2|\omega| + q$ .

The link between these two definitions is given by

Proposition (3.2.3)

$$S_\omega(\alpha)(x) = \langle x, S_\omega(\alpha) \rangle.$$

Proof:  $\langle x, S_\omega(\alpha) \rangle = \langle x, f_*(s_\omega(M) \cap o_M) \rangle,$

$$\begin{aligned}
&= \langle f^*(x), s_\omega(M) \cap o_M \rangle, \\
&= \langle f^*(x) \cdot s_\omega(M), o_M \rangle,
\end{aligned}$$

by 6.18 of Ch5 of [49].

Proposition (3.2.4)

$$1) \quad S_0(\alpha) = \mu(\alpha) \in H_p(X).$$

$$2) \quad \text{On } \Omega_*^U, \text{ if } \alpha = [M]$$

$$\begin{aligned}
S_\omega(\alpha) &= s_\omega[M], & 2|\omega| &= \text{dimension of } M, \\
&= 0, & \text{otherwise.}
\end{aligned}$$

Proof: Trivial.

We proceed now to study the behaviour of the  $S_\omega$ 's with respect to products. The fundamental result is

Proposition (3.2.5) (Thom)

If  $M$  and  $N$  are stably almost complex manifolds

$$s_\omega(M \times N) = \sum_{\omega_1 + \omega_2 = \omega} s_{\omega_1}(M) \times s_{\omega_2}(N) \in H^{2|\omega|}(M \times N),$$

where  $\times$  denotes the cohomology exterior product.

Proof: [37] Theorem 33, Corollary 3.

If we consider  $S_\omega$  as a natural transformation

$$S_\omega: \Omega_*^U(X) \longrightarrow H_*(X), \text{ of degree } -2|\omega|,$$

then writing  $\Delta S_\omega = \sum_{\omega_1 + \omega_2 = \omega} S_{\omega_1} \otimes S_{\omega_2}$  it is clear that

$$\Delta S_\omega: \Omega_*^U(X) \otimes \Omega_*^U(Y) \longrightarrow H_*(X) \otimes H_*(Y).$$

Lemma (3.2.6)

The following diagram commutes

$$\begin{array}{ccc}
 \Omega_*^U(X) \otimes \Omega_*^U(Y) & \xrightarrow{\times} & \Omega_*^U(X \times Y) \\
 \downarrow \Delta S_\omega & & \downarrow S_\omega \\
 H_*(X) \otimes H_*(Y) & \xrightarrow{\times} & H_*(X \times Y).
 \end{array}$$

Proof: Let  $\alpha = [M, f] \in \Omega_*^U(X)$ ,  $\beta = [N, g] \in \Omega_*^U(Y)$ , then

$$\begin{aligned}
 S_\omega(\alpha \times \beta) &= (f \times g)_*(s_\omega(M \times N) \cap o_{M \times N}), \\
 &= (f \times g)_*\left(\sum_{\omega_1 + \omega_2 = \omega} s_{\omega_1}(M) \times s_{\omega_2}(N) \cap (o_M \times o_N)\right), \\
 &= \sum_{\omega_1 + \omega_2 = \omega} f_*(s_{\omega_1}(M) \cap o_M) \times g_*(s_{\omega_2}(N) \cap o_N),
 \end{aligned}$$

using 6.2 and 6.13 of Ch5 of [49]. This proves the lemma.

Lemma (3.2.7)

$\Delta S_\omega$  factors uniquely as

$$\begin{array}{ccc}
 \Omega_*^U(X) \otimes \Omega_*^U(Y) & \xrightarrow{\Delta S_\omega} & H_*(X) \otimes H_*(Y) \\
 \downarrow & \nearrow \Delta S_\omega & \\
 \Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) & & 
 \end{array}$$

Proof: Just as in the proof of (3.1.1) it is sufficient

to show that if  $\alpha \otimes \varphi \otimes \beta \in \Omega_*^U(X) \otimes \Omega_*^U \otimes \Omega_*^U(Y)$  then

$$\Delta S_\omega(\alpha \varphi \otimes \beta) = \Delta S_\omega(\alpha \otimes \varphi \beta).$$

But using (3.2.6) with  $Y = \text{point}$  we have

$$\Delta S_\omega(\alpha \varphi \otimes \beta) = \sum_{\omega_1 + \omega_2 = \omega} s_{\omega_1}(\alpha \varphi) \otimes s_{\omega_2}(\beta),$$



$$\begin{aligned}
&= \sum_{\omega_1 + \omega_2 = \omega} \left( \sum_{\omega'_1 + \omega'_2 = \omega_1} S_{\omega'_1}(\alpha) \cdot S_{\omega'_2}(\varphi) \otimes S_{\omega_2}(\beta) \right) \\
&= \sum_{\omega_1 + \omega_2 + \omega_3 = \omega} S_{\omega_1}(\alpha) \cdot S_{\omega_2}(\varphi) \otimes S_{\omega_3}(\beta), \\
&= S_{\omega}(\alpha \otimes \varphi \beta),
\end{aligned}$$

by the reverse process, since  $S_{\omega_2}(\varphi) \in \mathbb{Z}$ .

Proposition (3.2.8)

$$\begin{array}{ccc}
\Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(Y) & \xrightarrow{\times} & \Omega_*^U(X \times Y) \\
\downarrow \Delta S_{\omega} & & \downarrow S_{\omega} \\
H_*(X) \otimes H_*(Y) & \xrightarrow{\times} & H_*(X \times Y) \quad \text{commutes.}
\end{array}$$

Proof: Follows immediately from the two lemmas.

Proposition (3.2.9)

If  $X$  is an  $H$ -space, so that  $\Omega_*^U(X)$  and  $H_*(X)$  are algebras then

$$\begin{array}{ccc}
\Omega_*^U(X) \otimes_{\Omega_*^U} \Omega_*^U(X) & \xrightarrow{\quad} & \Omega_*^U(X) \\
\downarrow \Delta S_{\omega} & & \downarrow S_{\omega} \\
H_*(X) \otimes H_*(X) & \xrightarrow{\quad} & H_*(X) \quad \text{commutes.}
\end{array}$$

Proof: Follows from (3.2.8), (3.1.3), and naturality.

Proposition (3.2.10)

If  $X$  satisfies the conditions of (3.1.4) and is also such that  $H_*(X)$  is torsion free, so that  $\Omega_*^U(X)$  and  $H_*(X)$  are coalgebras, then the following diagram commutes

$$\begin{array}{ccc}
\Omega_*^U(X) & \xrightarrow{\Delta} & \Omega_*^U(X) \otimes \Omega_*^U(X) \\
\downarrow S_\omega & & \downarrow \Delta S_\omega \\
H_*(X) & \xrightarrow{\Delta} & H_*(X) \otimes H_*(X).
\end{array}$$

Proof: As (3.2.9)

Expressing these propositions in terms of the characteristic numbers, the form in which we shall use them, we have

Corollary (3.2.11)

Let  $X$  be an  $H$ -space satisfying the conditions of Proposition (3.2.10), so that  $\Omega_*^U(X)$ ,  $H_*(X)$ , and  $H^*(X)$  are all Hopf algebras, then

1) If  $\alpha, \beta \in \Omega_*^U(X)$ ,  $x \in H^*(X)$ ,

$$S_\omega(\alpha\beta)(x) = (\Delta S_\omega)(\alpha \otimes \beta)(x).$$

2) If  $\alpha \in \Omega_*^U(X)$ ,  $x, y \in H^*(X)$ ,

$$(\Delta S_\omega)(\Delta\alpha)(x \otimes y) = S_\omega(\alpha)(xy).$$

Proof: 1) and 2) follow from (3.2.9) and (3.2.10) using (3.2.3) and duality.

Note: In the expansion of the equations of (3.2.11), most of the terms will vanish for dimensional reasons, as we shall see in specific examples below.

### §3.3 The Coefficient Ring

We recall here for later use some facts on the coefficient ring  $\Omega_*^U$ , its characteristic numbers and the Conner-Floyd homomorphism.

$\Omega_*^U \cong \mathbb{Z}[Y_1, Y_2, \dots]$ , a polynomial ring with one generator  $Y_i$  in each dimension  $2i$ , for all  $i > 0$ , [51]. We may choose the generators  $Y_i$  to be represented by Milnor base manifolds, [21] (41.1). That is, if  $i = p^k - 1$ , for some prime  $p$  and integer  $k$ , then

$$S_\omega(Y_i) \equiv 0 \pmod{p}, \text{ for all } \omega,$$

$$S_i(Y_i) = p,$$

and if  $i$  does not have this form

$$S_i(Y_i) = 1.$$

Here  $S_\omega(Y_i) \in H_0(\text{point}) \cong \mathbb{Z}$ .

There is no canonical way of choosing these generators, but, for any given  $i$ , we can compute the Chern numbers  $S_\omega$ , for a particular choice of  $Y_i$ . We shall need this information in the sequel for  $i = 1, 2, 3$ .

For  $i = 1, 2$ , we may take  $Y_i = [\mathbb{C}P^i]$ , then

$$S_1(Y_1) = 2,$$

$$S_2(Y_2) = 3,$$

and  $S_{11}(Y_2) = 3$ , see for example [37] XI.5.

Then, by (3.2.9), we have

$$\begin{aligned}
 S_2(Y_1^2) &= (\Delta S_2)(Y_1 \otimes Y_1), \\
 &= S_2(Y_1) \cdot S_0(Y_1) + S_0(Y_1) \cdot S_2(Y_1), \\
 &= 0, \\
 S_{11}(Y_1^2) &= (\Delta S_{11})(Y_1 \otimes Y_1), \\
 &= (S_1(Y_1))^2 \\
 &= 4.
 \end{aligned}$$

In dimension 6, since  $3 = 2^2 - 1$ , the situation is more complicated. We construct a representative for  $Y_3$  by considering a hypersurface of type  $(1,1)$  of  $\mathbb{CP}^2 \times \mathbb{CP}^2$ . This generalises for arbitrary  $Y_i$ , see [47] p. 235.

Specifically, let  $\pi_i: \mathbb{CP}^2 \times \mathbb{CP}^2 \longrightarrow \mathbb{CP}^2$ ,  $i=1,2$  be the projections, and let  $\xi$  be the dual of the canonical line bundle over  $\mathbb{CP}^2$ . Put  $\xi_i = \pi_i^* \xi$ ,  $i=1,2$  and  $\rho = \xi_1 \otimes \xi_2$ , then  $V \subset \mathbb{CP}^2 \times \mathbb{CP}^2$  is the submanifold dual to  $\rho$ .

Lemma (3.3.1)

$V$  has Chern numbers  $s_3[V] = -6$ ,

$$s_{12}[V] = 6,$$

$$\text{and } s_{111}[V] = 6.$$

Proof: Recall that  $H^*(\mathbb{CP}^2) \cong \mathbb{Z}[\alpha] / (\alpha^3)$ , where

$\alpha = c_1(\xi)$  and  $\langle \alpha^2, o_{\mathbb{CP}^2} \rangle = 1$ ;  $c(\mathbb{CP}^2) = (1+\alpha)^3 = 1+3\alpha+3\alpha^2$ .

Thus, if  $\alpha_i = \pi_i^* \alpha$ ,  $H^*(\mathbb{CP}^2 \times \mathbb{CP}^2) = \mathbb{Z}[\alpha_1, \alpha_2] / (\alpha_1^3, \alpha_2^3)$ ,

$\langle \alpha_1^2 \alpha_2^2, o_{\mathbb{CP}^2 \times \mathbb{CP}^2} \rangle = 1$ , and

$$c(\mathbb{CP}^2 \times \mathbb{CP}^2) = (1+\alpha_1)^3(1+\alpha_2)^3,$$

$$c(\rho) = (1+\alpha_1 + \alpha_2).$$

Then, if  $j: V \rightarrow \mathbb{CP}^2 \times \mathbb{CP}^2$  is the inclusion, [50] p.78,

$$\begin{aligned} c(V) &= j^* \left( \frac{(1+\alpha_1)^3(1+\alpha_2)^3}{(1+\alpha_1+\alpha_2)} \right), \\ &= j^*(1+(2\alpha_1+2\alpha_2)+(\alpha_1^2+5\alpha_1\alpha_2+\alpha_2^2) \\ &\quad + (3\alpha_1^2\alpha_2+3\alpha_1\alpha_2^2)). \end{aligned}$$

Thus  $s_{111}(V) = c_3(V) = j^*(3\alpha_1^2\alpha_2+3\alpha_1\alpha_2^2),$

$$s_{12}(V) = c_1(V)c_2(V)-3c_3(V) = j^*(3\alpha_1^2\alpha_2+3\alpha_1\alpha_2^2),$$

$$\begin{aligned} s_3(V) &= c_1(V)^3-3c_1(V)c_2(V)+3c_3(V), \\ &= j^*(-3\alpha_1^2\alpha_2-3\alpha_1\alpha_2^2). \end{aligned}$$

$$\text{Now } \langle j^*\alpha_1^2\alpha_2, o_V \rangle = \langle \alpha_1^2\alpha_2, j_*o_V \rangle,$$

but  $j_*o_V = c_1(\rho) \cap o_{\mathbb{CP}^2 \times \mathbb{CP}^2}$ , so

$$\begin{aligned} \langle j^*\alpha_1^2\alpha_2, o_V \rangle &= \langle \alpha_1^2\alpha_2, (\alpha_1+\alpha_2) \cap o_{\mathbb{CP}^2 \times \mathbb{CP}^2} \rangle, \\ &= \langle \alpha_1^2\alpha_2^2, o_{\mathbb{CP}^2 \times \mathbb{CP}^2} \rangle, \\ &= 1. \end{aligned}$$

Similarly  $\langle j^*\alpha_1\alpha_2^2, o_V \rangle = 1$ , and the result follows.

Now  $\mathbb{CP}^3$  has Chern numbers  $s_3[\mathbb{CP}^3] = 4,$

$$s_{12}[\mathbb{CP}^3] = 12,$$

$$\text{and } s_{111}[\mathbb{CP}^3] = 4.$$

So if we put  $Y'_3 = -[\mathbb{CP}^3] - [V] \in \Omega_6^U$ , we have

$$s_3(Y'_3) = 2,$$

$$s_{12}(Y'_3) = -18,$$

$$\text{and } s_{111}(Y'_3) = -10,$$

showing that  $Y'_3$  is a Milnor basis element. But for ease of computation, noting that

$$s_3(Y_1 Y_2) = s_3(Y_1^3) = 0,$$

$$s_{12}(Y_1 Y_2) = s_1(Y_1) s_2(Y_2) = 6,$$

$$s_{12}(Y_1^3) = s_1(Y_1) s_2(Y_1^2) = 0,$$

$$s_{111}(Y_1 Y_2) = s_1(Y_1) s_{11}(Y_2) = 6,$$

$$\text{and } s_{111}(Y_1^3) = s_1(Y_1)^3 = 8,$$

putting  $Y_3 = Y'_3 + 3Y_1 Y_2 - Y_1^3$ , we have

$$s_3(Y_3) = 2,$$

$$s_{12}(Y_3) = s_{111}(Y_3) = 0.$$

We will use this generator throughout in the computations below.

On the coefficient ring, the Conner-Floyd homomorphism

$$\mu: \Omega_*^U \longrightarrow K_*(\text{point}) = \mathbb{Z} \text{ is the map}$$

$$[M] \longmapsto \langle \text{todd}(M), o_M \rangle, \text{ see (3.1.6).}$$

In particular we note that

$$\mu(Y_1) = -1, \mu(Y_2) = 1, \mu(Y_3) = 0, \text{ see [29] 1.7.}$$

### §3.4 A Theorem on the Bordism of H-spaces

In this section we prove in a general form the fundamental theorem which we shall use later to compute the algebras  $\Omega_*^U(\Omega G)$  for various Lie groups  $G$ .

#### Theorem (3.4.1)

Let  $X$  be a homotopy commutative H-space, such that  $H_*(X)$  is torsion-free and even-dimensional. Suppose we have

$$H_*(X) \cong \frac{\mathbb{Z}[x_1, \dots, x_r]}{I},$$

where the  $x_i$  are homogeneous elements, and  $I$  is the ideal generated by elements  $f_j(x_1, \dots, x_r) = f_j(x)$   $j=1, \dots, k$  which are polynomials in the  $x_i$  of homogeneous dimension, chosen to be minimal generators of  $I$ .

$$\text{Then } \Omega_*^U(X) \cong \frac{\Omega_*^U[X_1, \dots, X_r]}{J},$$

where  $X_i \in \tilde{\Omega}_*^U(X)$  is an element such that  $\mu^{X_i} = x_i$   $i=1, \dots, r$  and  $J$  is the ideal generated by elements  $F_j(X)$   $j=1, \dots, k$

$$\text{where } F_j(X) = f_j(X) + \left\{ \begin{array}{l} \text{terms with positive dimensional} \\ \text{coefficients from } \Omega_*^U \end{array} \right\}.$$

Proof: The bordism spectral sequence with

$$E_{p,q}^2 = H_p(X; \Omega_q^U)$$

and limit  $\Omega_*^U(X)$ , identifies  $\mu$  with the edge-homomorphism

$$\Omega_n^U(X) \longrightarrow E_{n,0} \subset E_{n,0}^2 = H_n(X).$$

The spectral sequence collapses, for dimensional reasons, showing that  $\mu$  is an epimorphism and

$$\text{Ker } \mu = \text{Im} \{ \tilde{\Omega}_*^U \otimes \Omega_*^U(X) \longrightarrow \Omega_*^U(X) \},$$

where  $\tilde{\Omega}_*^U = \text{Ker} \{ \Omega_*^U \xrightarrow{\mu} H_*(\text{point}) = \mathbb{Z} \} = \bigoplus_{n>0} \Omega_n^U$ .

Choose homogeneous elements  $X_i \in \tilde{\Omega}_*^U(X)$  such that  $\mu X_i = x_i$ , we may do this since  $\mu$  is onto. Set

$$A = \Omega_*^U[X_1, \dots, X_r],$$

as graded algebra over  $\Omega_*^U$ , and define  $\varphi: A \longrightarrow \Omega_*^U(X)$  by using the Pontrjagin product in  $\Omega_*^U(X)$ , and making  $\Omega_*^U$ -linear.

Lemma (3.4.2)

$\varphi$  is an epimorphism.

Proof: By induction on dimension. It is trivial in dimension zero. Assume now that  $\varphi$  is an epimorphism in all dimensions less than  $n$ . Let  $\alpha \in \Omega_n^U(X)$ ,  $\mu\alpha \in H_n(X)$  can be written as a polynomial  $f_\alpha(x)$  in the  $x_i$ , and  $\mu(\alpha - f_\alpha(X)) = 0$ . Clearly  $f_\alpha(X)$  is in the image of  $\varphi$ , so it is sufficient to prove that  $\text{Ker } \mu \subset \text{Im } \varphi$ . Suppose then that  $\mu\alpha = 0$ , then  $\alpha$  has an expression as a sum of elements  $\pi\beta$ , where  $\pi \in \Omega_m^U$   $m>0$ ,  $\beta \in \Omega_{n-m}^U(X)$ ; but, by the induction hypothesis  $\beta \in \text{Im } \varphi$ , hence  $\pi\beta \in \text{Im } \varphi$ , and so  $\alpha \in \text{Im } \varphi$ . This proves the lemma.



Now consider the elements  $f_j(X) \in \Omega_*^U(X)$ . Since

$$f_j(X) = f_j(x) = 0 \in H_*(X),$$

using the lemma we can write  $f_j(X) = g_j(X)$  as a

polynomial in the  $X_i$  with coefficients in  $\hat{\Omega}_*^U$ , then take

$F_j = f_j - g_j$ , so that  $F_j(X) = 0 \in \Omega_*^U(X)$ . Let  $J$  be the

ideal in  $A$  generated by the  $F_j(X)$ , then  $\varphi$  factors

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A/J \longrightarrow 0, \\ & \searrow \varphi & \swarrow \varphi' \\ & \Omega_*^U(X) & \end{array}$$

and since  $\varphi$  is onto so is  $\varphi'$ .

#### Lemma (3.4.3)

$J \subset A$  is a direct summand as  $\mathbb{Z}$ -submodule.

Proof: We consider the ring homomorphism

$$\mu: A \longrightarrow \mathbb{Z}[x_1, \dots, x_r],$$

which sends  $X_i$  to  $x_i$ , and  $Y_i$  to 0. We can write

$$A = B \oplus \text{Ker} \mu \quad (3.4.4)$$

as  $\mathbb{Z}$ -module, where  $B$  is generated over  $\mathbb{Z}$  by monomials in the  $X_i$ .

$J$  is generated by the  $F_j$  over  $A$ , let  $C$  be the  $\mathbb{Z}$ -submodule generated by elements of the form  $m(X)F_j$ , where  $m(X)$  is a monomial in the  $X_i$ .

Since  $X$  is an H-space,  $H_*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_r] / I$  is a Hopf algebra with commutative diagonal and hence

is a polynomial algebra, [39]; it follows that each  $f_j$  has the form  $n_j x_{i_j} + \{\text{terms not involving } x_{i_j}\}$ , where  $n_j \in \mathbb{Z}$ . Thus since the  $f_j$  are minimal generators  $I$  has a  $\mathbb{Z}$  basis  $\{m(x)f_j\}$ . This shows then that we have

$$J = C \oplus (J \cap \text{Ker} \mu). \quad (3.4.5)$$

Now assume for an induction that  $i: J \subset A$  is split in dimensions less than  $n$ . In dimension  $n$ , with respect to the decompositions (3.4.4) and (3.4.5)  $i$  has the form

$$\begin{pmatrix} f & 0 \\ g & h \end{pmatrix}.$$

Now  $f: C \rightarrow B$  is essentially the inclusion  $I \subset \mathbb{Z}[x_1, \dots, x_r]$  in dimension  $n$ , which, since  $H_*(X)$  is torsion-free, is split, by  $p$  say. While  $h: (J \cap \text{Ker} \mu) \rightarrow \text{Ker} \mu$  is given from lower dimensional inclusions of  $J$  in  $A$  by multiplication by positive dimensional terms of  $\Omega_*^U$ ;  $h$  is thus split by  $q$  say, under the induction hypothesis.

$$\begin{aligned} \text{So } \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} f & 0 \\ g & h \end{pmatrix} &= \begin{pmatrix} pf & 0 \\ qg & qh \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ qg & I \end{pmatrix}: J \oplus (J \cap \text{Ker} \mu) \rightarrow \end{aligned}$$

which is an isomorphism showing that  $i$  is split in dimension  $n$ . The proof of the lemma is completed by noticing that

a basis for the induction is provided by the case

$J \cap \text{Ker } \mu = 0$  when  $h$  and  $q$  are vacuous.

Returning to the proof of the theorem, the spectral sequence show us that as  $\mathbb{Z}$ -module  $\Omega_*^U(X)$  is isomorphic to  $H_*(X) \otimes \Omega_*^U = \Omega_*^U[x_1, \dots, x_r] / (I \otimes \Omega_*^U)$ .

Since  $I$  and  $J$  have the same number of generators in the same dimensions, it is clear that

$$\text{rank}_{\mathbb{Z}} (I \otimes \Omega_*^U)_n = \text{rank}_{\mathbb{Z}} (J)_n, \text{ for each } n.$$

Thus  $\Omega_*^U(X)$  and  $A/J$  are both free  $\mathbb{Z}$ -modules, (3.4.3), of the same rank in each dimension, and so  $\varphi'$  being an epimorphism must be an isomorphism. This completes the proof of the theorem.

Suppose we can find suitable generators  $X_i$ , in the notation of the theorem, in such a form that we can calculate their characteristic numbers. If  $F_j = f_j - g_j$ , since  $F_j = 0$  in  $\Omega_*^U(X)$ , we have  $S_\omega(F_j)(x) = 0$ , for each partition  $\omega$ , and  $x \in H^*(X)$ . Thus we have equations

$$S_\omega(g_j)(x) = S_\omega(f_j)(x),$$

which are sufficient in general to determine the  $g_j$ .

$H_*(X)$  being torsion-free,  $H_*(X)$  and  $\Omega_*^U(X)$  will be Hopf algebras. Suppose the diagonal in  $H_*(X)$  is given as

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i + \sum_t x'_{i,t} \otimes x''_{i,t},$$

where  $x'_{i,t}$  and  $x''_{i,t}$  are elements of  $\tilde{H}_*(X)$ .

Then, since  $\mu: \Omega_*^U(X) \longrightarrow H_*(X)$  is a coalgebra homomorphism, by naturality of the coalgebra structures, we must have in  $\Omega_*^U(X)$

$$\Delta X_i = X_i \otimes 1 + 1 \otimes X_i + \sum_t X'_{i,t} \otimes X''_{i,t} + \{\text{extra terms}\},$$

where  $X'_{i,t}$  and  $X''_{i,t}$  are elements mapped to  $x'_{i,t}$  and  $x''_{i,t}$

by  $\mu$ , and the extra terms have the form

$$\pi \alpha \otimes \beta \quad \pi \in \tilde{\Omega}_*^U, \quad \alpha \in \hat{\Omega}_*^U(X), \quad \beta \in \tilde{\Omega}_*^U(X).$$

(Recall  $\Delta X_i$  lies in  $\Omega^U(X) \otimes_{\Omega_*^U} \Omega^U(X)$ .)

Then if we have representatives suitable for computing characteristic numbers the equations

$$(\Delta S_\omega)(\Delta X_i)(x \otimes y) = S_\omega(X_i)(xy)$$

of Corollary (3.2.11) 2) will provide enough information to compute the extra terms. We can also use the fact that the diagonal is associative to obtain a check on the characteristic number calculations.

### §3.5 Homology of Loop Spaces of Lie Groups

In the following sections we will apply the ideas of §3.4 to obtain results on the Unitary bordism of the loop spaces of Lie groups, but first we must recall the

results of [15] and [14] on their homology, and see how these provide us with bordism generators.

We suppose  $G$  is a compact, 1-connected, simple Lie group of rank  $r$ , and that  $T \subset G$  is a maximal torus. Let  $R$  be the set of roots of  $G$ . Each  $\theta \in R$  is an element of  $\mathfrak{t}^*$  the dual of the Lie algebra of  $T$ . We assume given an ordering on the roots with respect to which  $\mu$  is the dominant root. Let  $R_+$  be the set of positive roots.

The set  $\mathcal{O}$  of oriented singular planes in  $\mathfrak{t}$  is the set of

$$p = (\theta_p, n_p) = \{x \in \mathfrak{t} : \theta_p(x) = n_p\},$$

where  $\theta_p \in R_+$ ,  $n_p \in \mathbb{Z}$ .

The diagram  $D(G)$  is the union of all  $p \in \mathcal{O}$ ; the infinitesimal diagram  $D'(G)$  is the union of the  $p \in \mathcal{O}$  such that  $n_p = 0$ . The components of  $\mathfrak{t} - D(G)$  are the cells; the components of  $\mathfrak{t} - D'(G)$  are the chambers. Let  $\mathcal{F}$  be the fundamental chamber in which all the  $\theta \in R_+$  take positive values.

Theorem (3.5.1) (Bott and Samelson) (Petrie)

1) For each cell  $\Delta \subset \mathcal{F}$ , there is defined a complex manifold  $\Gamma_\Delta$  and a map  $f_\Delta : \Gamma_\Delta \longrightarrow \Omega G$ , such that the set

$$\{f_{\Delta*}(o_{\Gamma_\Delta}) : \Delta \subset \mathcal{F}\}$$

is a basis for the free group  $H_*(\Omega G)$ .

2) Let  $\{p_1, \dots, p_t\}$  be the set of oriented planes in ,  $p_j = (\theta_j, n_j)$ , which a line segment from  $\Delta$  to 0 crosses, in that order.  $\Gamma_\Delta$  is  $2t$ -dimensional and

$$H^*(\Gamma_\Delta) \cong \frac{\mathbb{Z}[x_1, \dots, x_t]}{(\rho_1, \dots, \rho_t)}, \text{ where } x_i \in H^2(\Gamma_\Delta),$$

$$\text{and } \rho_k = x_k^2 + \sum_{j=1}^{k-1} n(\theta_j, \theta_k) x_j x_k,$$

here, if  $\theta, \varphi \in R$ ,  $n(\theta, \varphi)$  is the Cartan integer  $\frac{2(\theta, \varphi)}{(\varphi, \varphi)}$ ,  $(\cdot, \cdot)$  being the Killing form.

3) The total Chern class of  $\Gamma_\Delta$  is

$$c(\Gamma_\Delta) = \prod_{i=1}^t (1 + 2x_i + \sum_{j=1}^{i-1} n(\theta_j, \theta_i) x_j).$$

4) There is a canonical generator  $\eta \in H^2(\Omega G)$  and

$$f_\Delta^*(\eta) = \sum_{i=1}^t n_i \left( \frac{(\mu, \mu)}{(\theta_i, \theta_i)} \right) x_i.$$

Proof: For 1), 2), and 4) see [15] 9.1, 4.2, and 11.2 of Chapter III. For 3), [40] 6.5.

The theorem shows us that we may use the  $[\Gamma_\Delta, f_\Delta]$  as complex bordism generators. The Chern classes may be computed using 3), and 4) provides some information on the map  $f_\Delta$ , but not in general enough. Also there is no straitforward connection between the  $\Gamma_\Delta$ -generators of [15] and the polynomial generators of [14].

In [14] Bott completes his study of  $H_*(\Omega G)$  by determining the multiplicative and coalgebra structures. His method is the 'generating variety' approach which we considered in chapter 2. The fundamental result is

Theorem (3.5.2)

There is a maximal rank subgroup  $U \subset G$  and a map

$$g: G/U \longrightarrow \Omega G$$

such that  $\text{Im}\{g_*: H_*(G/U) \longrightarrow H_*(\Omega G)\}$  generates  $H_*(\Omega G)$  multiplicatively.

Proof: [14].

Bott then obtains an expression of the form

$$H_*(G) \cong \frac{\mathbb{Z}[x_1, \dots, x_s]}{I} \quad x_i \text{ even-dimensional.}$$

Now the space  $G/U$  is a complex manifold, in fact it is algebraic, [44]. The results of Borel-Hirzebruch give methods for computing its Chern classes. (3.5.2) then makes it easy to compute the map  $g_*: H_*(G/U) \longrightarrow H_*(\Omega G)$ , and hence the characteristic numbers of the bordism element  $[G/U, g]$ . We will also use various submanifolds of  $G/U$  as bordism generators in particular cases, as well as generators induced by the inclusion of a subgroup of  $G$ .

### §3.6 Unitary and Symplectic Groups

Since  $H_*(G)$  is torsion-free for  $G = \mathrm{SU}(n+1)$ ,  $\mathrm{Sp}(n)$   $H_*(\Omega\mathrm{SU}(n+1))$  and  $H_*(\Omega\mathrm{Sp}(n))$  are polynomial algebras. In fact we have

$$H_*(\Omega\mathrm{SU}(n+1)) = \mathbb{Z}[u_1, u_2, \dots, u_n],$$

$$H_*(\Omega\mathrm{Sp}(n)) = \mathbb{Z}[u_1, u_3, \dots, u_{2n-1}],$$

where dimension of  $u_i = 2i$ .

Thus letting  $X = \Omega\mathrm{SU}(n+1)$  or  $\Omega\mathrm{Sp}(n)$  in Theorem (3.4.1), the ideal  $I$  is the zero ideal and so also is  $J$ . We have proved therefore

#### Theorem (3.6.1)

$$1) \quad \Omega_*^U(\Omega\mathrm{SU}(n+1)) = \Omega_*^U[u_1, u_2, \dots, u_n].$$

$$2) \quad \Omega_*^U(\Omega\mathrm{Sp}(n)) = \Omega_*^U[u_1, u_3, \dots, u_{2n-1}],$$

where dimension of  $U_i = 2i$  in both cases.

#### Corollary (3.6.2)

$K_*(\Omega\mathrm{SU}(n+1))$  and  $K_*(\Omega\mathrm{Sp}(n))$  are polynomial algebras over  $\mathbb{Z}$  with  $n$  generators in  $K_0$ .

Proof: Conner-Floyd isomorphism.

Now for  $\Omega\mathrm{SU}(n+1)$  the generating variety is complex projective  $n$ -space  $\mathbb{CP}^n$ , and the generating map

$$g: \mathbb{CP}^n \longrightarrow \Omega\mathrm{SU}(n+1)$$

has a particularly simple form, see [25] p.76.



Lemma (3.6.3)

Put  $H^*(\mathbb{C}P^n) = \frac{\mathbb{Z}[\alpha]}{(\alpha^{n+1})}$   $\alpha \in H^2(\mathbb{C}P^n)$  and let

$\{1, y_1, \dots, y_n\}$  be a basis of  $H_*(\mathbb{C}P^n)$  dual to  $\{1, \alpha, \dots, \alpha^n\}$ ,

then  $g_* y_i = u_i$ .

Proof: This is the definition of the  $u_i$  in 8.1 of [14], compare (2.6.1).

The lemma shows that if we consider  $\mathbb{C}P^i \subset \mathbb{C}P^n$  and  $g|_{\mathbb{C}P^i} = g_i: \mathbb{C}P^i \longrightarrow \Omega SU(n+1)$ , then  $g_{i*}(o_{\mathbb{C}P^i}) = u_i$ , which proves

Corollary (3.6.4)

In theorem (3.6.1) we may take  $U_i = [\mathbb{C}P^i, g_i^{-1}]$ .

Notation If  $[M, f] \in \Omega_*^U(X)$ , where  $X$  is a based space, we write  $[M, f-1] = [M, f] - [M, 1]$  where  $1: M \longrightarrow X$  is the trivial map, then  $[M, f-1] \in \tilde{\Omega}_*^U(X)$ .

From (2.6.1) we know that, with respect to suitable polynomial generators,  $\hat{u}_i$ ,  $K_*(\Omega SU(n+1))$  has diagonal of the form

$$\Delta \tilde{u}_k = \sum_{i+j=k} \hat{u}_i \otimes \hat{u}_j.$$

In fact the methods of chapter 2 should extend to prove a similar result for  $\Omega_*^U(\Omega SU(n+1))$ , but the following proposition shows that the generators of (3.6.4) are

not suitable.

Proposition (3.6.5)

With the notation of (3.6.4) we have

$$\Delta U_3 = U_3 \otimes 1 + U_2 \otimes U_1 - Y_1 U_1 \otimes U_1 + U_1 \otimes U_2 + 1 \otimes U_3.$$

Proof: By the remarks at the end of §3.4, we can set

$$\Delta U_3 = U_3 \otimes 1 + U_2 \otimes U_1 + U_1 \otimes U_2 + 1 \otimes U_3 + k Y_1 U_1 \otimes U_1,$$

for some  $k \in \mathbb{Z}$ . Let  $\eta \in H^2(\Omega SU(n+1))$  be dual to  $u_1$ . By

$$(3.2.11) \ 2) \quad S_1(U_3)(\eta^2) = (\Delta S_1)(\Delta U_3)(\eta \otimes \eta).$$

$$\text{Now } S_1(U_3)(\eta^2) = \langle s_1(\mathbb{CP}^3).g^*(\eta^2), o_{\mathbb{CP}^3} \rangle = 4.$$

$$\begin{aligned} (\Delta S_1)(\Delta U_3)(\eta \otimes \eta) &= (S_1 \otimes S_0)(U_2 \otimes U_1 + k Y_1 U_1 \otimes U_1)(\eta \otimes \eta) \\ &\quad + (S_0 \otimes S_1)(U_1 \otimes U_2)(\eta \otimes \eta), \end{aligned}$$

all the other terms drop out for dimensional reasons.

$$S_1(U_2)(\eta) = \langle s_1(\mathbb{CP}^2).g^*(\eta), o_{\mathbb{CP}^2} \rangle = 3,$$

$$S_0(U_1)(\eta) = \langle \eta, \mu U_1 \rangle = 1,$$

$$\begin{aligned} S_1(Y_1 U_1)(\eta) &= (\Delta S_1)(Y_1 \otimes U_1)(\Delta \eta), \quad (3.2.11) \ 1), \\ &= S_1(Y_1)(1) \cdot S_0(U_1)(\eta), \\ &= S_1(Y_1) = 2. \end{aligned}$$

Thus  $4 = 6 + 2k$ , giving  $k = -1$ .

Note: By taking  $U'_3 = U_3 + Y_1 U_2$ , which can replace  $U_3$  as a polynomial generator, we can eliminate the cross-term  $Y_1 U_1 \otimes U_1$ .

The diagonal in  $H_*(\Omega Sp(n))$  is not known in general.

For  $n = 2$ , [14] 9.1,  $H_*(\Omega Sp(2)) = \mathbb{Z}[u_1, u_3]$  with

$$\Delta u_1 = u_1 \otimes 1 + 1 \otimes u_1,$$

$$\Delta u_3 = u_3 \otimes 1 + u_1^2 \otimes u_1 + u_1 \otimes u_1^2 + 1 \otimes u_3.$$

The generating variety is  $Sp(2)/U(2)$ .

Theorem (3.6.6)

$$\Omega_*^U(\Omega Sp(2)) = \Omega_*^U[U_1, U_3],$$

where we may take  $U_3 = [Sp(2)/U(2), g^{-1}]$ .

$U_1$  is primitive and

$$\Delta U_3 = U_3 \otimes 1 + U_1^2 \otimes U_1 + 3Y_1 U_1 \otimes U_1 + U_1 \otimes U_1^2 + 1 \otimes U_3.$$

Proof: By [9]  $H^*(\frac{Sp(2)}{U(2)}) = \frac{\mathbb{Z}[u, v]}{(u^2 - 2v, v^2)}$   $\dim u = 2$ ,  
 $\dim v = 4$ .

If we let  $\{1, \sigma_1, \sigma_2, \sigma_3\}$  be a basis of  $H^*(\frac{Sp(2)}{U(2)})$  dual to  $\{1, u, v, uv\}$ , then  $g_* \sigma_1 = u_1$ ,  $g_* \sigma_2 = u_1^2$ ,  $g_* \sigma_3 = u_3$ .

The total Chern class of  $Sp(2)/U(2)$  is

$$c(\frac{Sp(2)}{U(2)}) = 1 + 3u + 8v + 4uv, \text{ by [12] 16.4, and}$$

since  $\langle c_3, o_{\frac{Sp(2)}{U(2)}} \rangle = \chi(\frac{Sp(2)}{U(2)}) = 4$ , it is clear that

$\sigma_3 = o_{\frac{Sp(2)}{U(2)}}$ . Thus we can take  $U_3 = [\frac{Sp(2)}{U(2)}, g^{-1}]$ .

Write  $\Delta U_3 = U_3 \otimes 1 + U_1^2 \otimes U_1 + U_1 \otimes U_1^2 + 1 \otimes U_3 + kY_1 U_1 \otimes U_1$ ,

for some  $k \in \mathbb{Z}$ . Let  $\eta$  be dual to  $u_1$ . We use the identity

$$S_1(U_3)(\eta^2) = (\Delta S_1)(\Delta U_3)(\eta \otimes \eta).$$

We have 
$$S_1(U_3)(\eta^2) = \langle c_1 \cdot g^*(\eta^2), \sigma_3 \rangle,$$

$$= \langle 3u \cdot u^2, \sigma_3 \rangle = 6.$$

$$S_1(U_1^2)(\eta) = 2S_1(U_1)(1) \cdot S_0(U_1)(\eta) = 0.$$

$$S_1(Y_1 U_1)(\eta) = S_1(Y_1) = 2.$$

Thus  $6 = 2k$ , and  $k = 3$ .

Corollary ( 3.6.7)

$$K_*(\Omega Sp(2)) = \mathbb{Z}[\tilde{u}_1, \tilde{u}_3], \quad \tilde{u}_i \in K_0,$$

$$\Delta \tilde{u}_1 = \tilde{u}_1 \otimes 1 + 1 \otimes \tilde{u}_1,$$

$$\Delta \tilde{u}_3 = \tilde{u}_3 \otimes 1 + \tilde{u}_1^2 \otimes \tilde{u}_1 - 3\tilde{u}_1 \otimes \tilde{u}_1 + \tilde{u}_1 \otimes \tilde{u}_1^2 + 1 \otimes \tilde{u}_3.$$

(This is in agreement with Theorem (2.6.5) if we put

$$\tilde{u}_1 = u, \quad \tilde{u}_3 = v - 2u^2.)$$

Proof: Conner-Floyd isomorphism.

Note: In contrast to the unitary case the generator  $U_3$  cannot be changed to eliminate the cross-term  $Y_1 U_1 \otimes U_1$ , since the only element having  $U_1 \otimes U_1$  as a cross-term is  $U_1^2$ , but  $\Delta U_1^2 = U_1^2 \otimes 1 + 2U_1 \otimes U_1 + 1 \otimes U_1^2$  and the coefficient of  $Y_1 U_1 \otimes U_1$  in  $\Delta U_3$  is odd.

In a Hopf algebra  $A$ , this property of non-primitive generatability is expressed in a coordinate free way by the map  $PA \longrightarrow \tilde{A} \longrightarrow QA$  where  $PA$  is the primitive module and  $QA$  is the module of indecomposables. All the

Hopf algebras we consider are torsion-free over the ground ring so the map  $PA \longrightarrow QA$  is a monomorphism, its cokernel gives a measure of how far  $A$  is from being primitively generated.

### Corollary (3.6.8)

- 1)  $\text{Coker}\{PH_*(\Omega Sp(2)) \longrightarrow QH_*(\Omega Sp(2))\} \cong \mathbb{Z}_3$  in dimension 6.
- 2)  $\text{Coker}\{P\Omega_*^U(\Omega Sp(2)) \longrightarrow Q\Omega_*^U(\Omega Sp(2))\}$  is the graded  $\Omega_*^U$ -module generated by one element of order 6 in dimension 6.
- 3)  $\text{Coker}\{PK_*(\Omega Sp(2)) \longrightarrow QK_*(\Omega Sp(2))\} \cong \mathbb{Z}_6$ .

Proof: We prove 2), 1) is even easier, and 3) follows from 2).

We find easily that  $P\Omega_*^U(\Omega Sp(2))$  is the free  $\Omega_*^U$ -module generated by  $\{U_1, (6U_3 - 2U_1^3 - 9Y_1U_1^2)\}$ .  $Q\Omega_*^U(\Omega Sp(2))$  is generated by the classes of  $U_1$  and  $U_3$  modulo decomposable elements. Hence the result.

### 3.7 The Orthogonal Groups

The generating variety of  $\Omega Spin(n+2)$  is

$$\frac{SO(n+2)}{SO(2) \times SO(n)} = Q_n.$$

$Q_n$  can be considered as the hypersurface given by

$z_0^2 + z_1^2 + \dots + z_{n+1}^2 = 0$  in  $\mathbb{CP}^{n+1}$ .  $Q_{2m}$  contains two distinct subspaces  $\mathbb{CP}^m$  and  $'\mathbb{CP}^m$ ,  $\mathbb{CP}^m \cap '\mathbb{CP}^m = \mathbb{CP}^{m-1}$ .

$Q_{2m-1}$  contains a subspace  $\mathbb{CP}^{m-1}$ , [30] ChXIII §6 theorem I.

$H_*(Q_{2m-1})$  is generated by  $\{1, \sigma_1, \dots, \sigma_{2m-1}\}$   $\dim \sigma_i = 2i$ , where for  $i \leq m-1$  we may take  $\sigma_i = \text{image of } o_{\mathbb{CP}^i} \text{ under the inclusion } \mathbb{CP}^i \subset \mathbb{CP}^{m-1} \subset Q_{2m-1}$ .

$H_*(Q_{2m})$  is generated by  $\{1, \sigma_1, \dots, \sigma_{m-1}, \sigma_m^+, \sigma_m^-, \dots, \sigma_{2m}\}$  where we may take the  $\sigma_i$ ,  $i \leq m-1$  to come from  $\mathbb{CP}^i$  as for  $Q_{2m-1}$ , and  $\sigma_m^{\pm}$  to come from  $\mathbb{CP}^m$  and  $'\mathbb{CP}^m$ .

Bott's result is then that if  $a = \lfloor \frac{m}{2} \rfloor$ ,  $b = 2a+1$ ,

$$H_*(\Omega \text{Spin}(2m+1)) = \frac{\mathbb{Z}[u_1, u_2, \dots, u_{m-1}, u_b, u_{b+2}, \dots, u_{2m-1}]}{I}$$

where  $I$  is generated by

$$f_i = u_i^2 - 2u_{i-1}u_{i+1} + \dots + (-1)^{i-1}2u_1u_{i-1} + (-1)^i2u_{2i} \\ i=1, \dots, \lfloor \frac{m-1}{2} \rfloor, \text{ and } g_*\sigma_i = u_i \quad i=1, \dots, m-1.$$

$$H_*(\Omega \text{Spin}(2m+2)) = \frac{\mathbb{Z}[u_1, u_2, \dots, u_{m-1}, v_m, u_b, u_{b+2}, \dots, u_{2m-1}]}{I}$$

where  $I$  is the same ideal as in the odd case, and as before

$$g_*\sigma_i = u_i \quad i=1, \dots, m-1.$$

It follows then that we have

$$\Omega_*^U(\Omega \text{Spin}(2m+1)) = \frac{\Omega_*^U[U_1, \dots, U_{m-1}, U_b, U_{b+2}, \dots, U_{2m-1}]}{J},$$

$$\Omega_*^U(\Omega\text{Spin}(2m+2)) = \frac{\Omega_*^U[U_1, \dots, U_{m-1}, V_m, U_b, U_{b+2}, \dots, U_{2m-1}]}{J},$$

and, if we let  $g_i = g|_{\mathbb{CP}^i: \mathbb{CP}^i \rightarrow \Omega\text{Spin}(n)}$ , we can take

$$U_i = [\mathbb{CP}^i, g_i^{-1}] \quad i=1, \dots, m-1.$$

$J$  is generated by  $F_i \quad i=1, \dots, [\frac{m-1}{2}]$ , and

$$F_i = f_i(U) + \sum_{j=1}^{2i-1} V_{i, 2i-j} U_j + \{\text{decomposable terms}\} \quad (3.7.1)$$

where  $V_{i, 2i-j} \in \Omega_{2i-j}^U$ .

Proposition (3.7.2) (Petrie, [40] 2.1)

If  $r$  is odd

$$V_{i,r} = (-1)^{i+1} (2i+1) Y_r + \left\{ \begin{array}{l} \text{decomposable elements} \\ \text{in } \Omega_*^U \end{array} \right\}$$

if  $r = 2^k - 1$  for some  $k$ ,

$$V_{i,r} = (-1)^{i+1} 2(2i+1) Y_r + \left\{ \begin{array}{l} \text{decomposable elements} \\ \text{in } \Omega_*^U \end{array} \right\}$$

otherwise.

Proof: By the results on  $\Omega_*^U$  and the Milnor generators  $Y_j$ , see §3.3, it is clearly sufficient to prove

$$S_r(V_{i,r}) = (-1)^{i+1} 2(2i+1).$$

Choose a basis for  $H_{2i-r}(\Omega\text{Spin}(n+2))$  containing the element  $u_{2i-r}$ . Form a dual basis for  $H^{2i-r}(\Omega\text{Spin}(n+2))$  and let  $\eta_{2i-r}$  be dual to  $u_{2i-r}$ . Since  $r$  is odd, no multiple of  $u_{2i-r}$  is decomposable and so  $\eta_{2i-r}$  is primitive. We will evaluate  $S_r(F_i)(\eta_{2i-r}) = 0$ .

Since  $\eta_{2i-r}$  is primitive  $S_r(\alpha)(\eta_{2i-r}) = 0$  for any decomposable element  $\alpha \in \Omega_*^U(\Omega\text{Spin}(n+2))$ .

Thus on applying  $S_r(\cdot)(\eta_{2i-r})$  to the expansion (3.7.1) we get

$$0 = (-1)^i 2S_r(U_{2i})(\eta_{2i-r}) + \sum_{j=1}^{2i-1} S_r(V_{i,2i-j}U_j)(\eta_{2i-r}).$$

$$\begin{aligned} \text{Now } S_r(V_{i,2i-j}U_j)(\eta_{2i-r}) &= (\Delta S_r)(V_{i,2i-j} \otimes U_j)(\Delta \eta_{2i-r}), \\ &= S_r(V_{i,2i-j})(1) \cdot S_0(U_j)(\eta_{2i-r}), \\ &\begin{cases} = S_r(V_{i,r}) & 2i-j = r \\ = 0 & 2i-j \neq r. \end{cases} \end{aligned}$$

$U_{2i} = [\mathbb{CP}^{2i}, g_{2i}^{-1}]$  so we can calculate that

$$S_r(U_{2i})(\eta_{2i-r}) = 2i+1.$$

Hence the result.

Corollary (3.7.3)

$$V_{i,1} = (-1)^{i+1} (2i+1) Y_1$$

Proof: Immediate

Using these results and additional computations we can prove

Theorem (3.7.4)

For  $\Omega_*^U(\Omega\text{Spin}(n))$   $n = 7, 8, 9, 10$  the ideal  $J$  is

$$(U_1^2 - 2U_2 + 3Y_1U_1),$$

for  $n = 11, 12, 13, 14$   $J$  is



$$(U_1^2 - 2U_2 + 3Y_1U_1, U_2^2 - 2U_1U_3 + 2U_4 - 5Y_1U_3 + Y_1U_1U_2 + 2(Y_1^2 + Y_2)U_2 + (Y_1^3 - 3Y_1Y_2 - 5Y_3)U_1).$$

This method would yield the multiplicative structure of  $\Omega_*^U(\Omega\text{Spin}(n))$  for any given  $n$ , if one was prepared to do enough arithmetic.

Note from Bott's results [14] 9,10 that under the inclusion  $i: \Omega\text{Spin}(n) \longrightarrow \Omega\text{Spin}(n+1)$  the homology generators  $u_j$  are mapped into themselves. Thus if

$$[M_j, f_{j-1}] = U_j \in \Omega_{2j}^U(\Omega\text{Spin}(n)), \text{ then}$$

$$[M_j, i.f_{j-1}] = i_*U_j = U_j \in \Omega_{2j}^U(\Omega\text{Spin}(n+1)).$$

In this way we can get bordism representatives for the classes  $U_b, \dots, U_{2m-3}$ , and obtain

### Corollary (3.7.5)

The relations of (3.7.4) hold in  $\Omega_*^U(\Omega\text{Spin}(n))$  for all appropriate  $n$ .

### Theorem (3.7.6)

1) In  $\Omega_*^U(\Omega\text{Spin}(n))$   $n \geq 5$

$$\begin{aligned} \Delta U_3 &= U_3 \otimes 1 + U_1^2 \otimes U_1 + 3Y_1 U_1 \otimes U_1 + U_1 \otimes U_1^2 + 1 \otimes U_3, \\ &= U_3 \otimes 1 + 2U_2 \otimes U_1 - 3Y_1 U_1 \otimes U_1 + 2U_1 \otimes U_2 + 1 \otimes U_3 \quad n \geq 7. \end{aligned}$$

2) In  $\Omega_*^U(\Omega\text{Spin}(n))$   $n \geq 7$

$$\begin{aligned}
U_5 = & U_5 \otimes 1 + U_1 U_3 \otimes U_1 - U_2^2 \otimes U_1 + U_3 \otimes U_2 + U_2 \otimes U_3 - U_1 \otimes U_2^2 + U_1 \otimes U_1 U_3 + 1 \otimes U_3 \\
& + Y_1 (U_3 \otimes U_1 - 6 U_1 U_2 \otimes U_1 - U_2 \otimes U_2 - 6 U_1 \otimes U_1 U_2 + U_1 \otimes U_3) \\
& + 7 Y_1^2 (U_2 \otimes U_1 + U_1 \otimes U_2) \\
& - (81 Y_1^3 - 2 Y_1 Y_2 + Y_3) (U_1 \otimes U_1).
\end{aligned}$$

Proof: 1) follows from (3.6.6) and (3.7.6). 2) is a calculation.

### §3.8 The Exceptional Lie Group $G_2$

We must first clear up the result of Bott, [14] 11, on  $H_*(\Omega G_2)$ , which contains certain errors. In particular the diagonal he gives is not associative, as may be easily checked.

The generating variety  $V$  for  $G_2$  is the space  $G_2/U(2)$ . One must be slightly careful here since there are two non-conjugate subgroups  $U(2) \subset G_2$ , given in the prescription of Borel-Siebenthal, [11], by the inclusion of the two nodes of the Dynkin diagram  $\begin{array}{c} \circ \\ \parallel \\ \circ \end{array}$  of  $G_2$  (notation of Bourbaki [16]). The case we wish to consider, see [14] 5, is when  $U(2)$  is given as the centraliser of the dual direction to a long root, that is the inclusion of the left hand node of the diagram.

As Bott suggests we may use the methods of [15] to compute  $H^*(G_2/T)$  and then  $H^*(G_2/U(2)) = H^*(G_2/T)^{W(U(2))}$ .

Somewhat less long-winded is to compute the composition

$$R(U(2)) \xrightarrow{\alpha} K^*(G_2/U(2)) \xrightarrow{\text{ch}} H^*(G_2/U(2); \mathbb{Q}),$$

compare the proof of (2.6.2) to which we refer for notation.

And then use the fact that, since  $H^*(G_2/U(2))$  is torsion-free, [13], every element of  $H^*(G_2/U(2))$  occurs as the beginning of a Chern character, [8] 2.5.

Lemma (3.8.1)

$$1) \quad H^{**}(B_T; \mathbb{Q}) = \mathbb{Q}[[x_1, x_2]] \quad \dim x_i = 2,$$

$W(G_2)$  is generated by  $s_1$  and  $s_2$  which act by

$$s_1: (x_1, x_2) \longmapsto (x_2 - x_1, x_2)$$

$$s_2: (x_1, x_2) \longmapsto (x_1, 3x_1 - x_2)$$

on  $H^2(B_T; \mathbb{Q})$ , extending multiplicatively to  $H^{**}(B_T; \mathbb{Q})$ .

Under the inclusion  $U(2) \subset G_2$  which we are considering

$W(U(2)) \subset W(G_2)$  is the subgroup generated by  $s_1$ .

$$2) \quad H^{**}(B_{U(2)}; \mathbb{Q}) = H^{**}(B_T; \mathbb{Q})^{W(U(2))} = \mathbb{Q}[[y_1, y_2]],$$

$$H^{**}(B_{G_2}; \mathbb{Q}) = H^{**}(B_T; \mathbb{Q})^{W(G_2)} = \mathbb{Q}[[z_1, z_2]],$$

where  $y_1 = x_2$ ,

$$y_2 = x_1^2 - x_1 x_2,$$

$$z_1 = 3x_1^2 - 3x_1 x_2 + x_2^2,$$

$$z_2 = 4x_1^6 - 12x_1^5 x_2 + 13x_1^4 x_2^2 - 6x_1^3 x_2^3 + x_1^2 x_2^4.$$

$$\begin{aligned} 3) \quad H^*(G_2/U(2); \mathbb{Q}) &= \frac{\mathbb{Q}[y_1, y_2]}{(z_1, z_2)}, \\ &= \frac{\mathbb{Q}[u]}{(u^5)}, \text{ where } u = y_1. \end{aligned}$$

Proof: 1) is standard, see for example [12] §18. 2) and 3) are a matter of calculation, using either elementary algebra or the algorithm of [26].

Proposition (3.8.2)

$H^*(G_2/U(2))$  has a basis  $\{1, u, \frac{u^2}{3}, \frac{u^3}{6}, \frac{u^4}{18}, \frac{u^5}{18}\}$  where  $u \in H^2(G_2/U(2))$  is a generator.

Proof:  $R(U(2)) = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_2^{-1}]$ , and as in (2.6.2) the image of the Chern character in  $H^*(G_2/U(2))$  is the subring generated by  $\text{ch}\alpha(\lambda_1)$  and  $\text{ch}\alpha(\lambda_2)$ .

$$\chi(\lambda_1) = \exp(x_1) + \exp(x_2 - x_1)$$

$$\chi(\lambda_2) = \exp(x_2).$$

It is now a simple calculation using the lemma that

$$\text{ch}\alpha(\lambda_1) = 2 + u + \frac{u^2}{6} - \frac{u^4}{9 \cdot 4!} - \frac{u^5}{9 \cdot 5!},$$

$$\text{ch}\alpha(\lambda_2) = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!}.$$

Then a basis for  $H^*(G_2/U(2))$  is provided by elements

occurring at the beginning of multiplicative combinations

of the elements  $a_1 = \text{ch}\alpha(\lambda_1) - 2$ ,  $a_2 = \text{ch}\alpha(\lambda_2) - 1$ .

Thus  $u$  generates  $H^2(G_2/U(2))$ . Since

$$a_2 - a_1 = \frac{u^2}{3} + \frac{u^3}{6} + \dots, \quad \frac{u^2}{3} \text{ generates } H^4.$$

$$a_2^2 - 3(a_2 - a_1) = \frac{u^3}{2} + \dots, \text{ thus } 2|u^3, \text{ but } 3|u^3$$

since  $3|u^2$ , and we see that  $\frac{u^3}{6}$  generates  $H^6$ . Similarly

for  $H^8$  and  $H^{10}$ , proving the proposition.

Theorem (3.8.3) (Bott)

$$H_*(\Omega G_2) = \frac{\mathbb{Z}[u, v, w]}{(u^2 - 2v)} \quad \begin{array}{l} \dim u = 2, \\ \dim v = 4, \\ \dim w = 10, \end{array}$$

with diagonal

$$u = u \otimes 1 + 1 \otimes u,$$

$$v = v \otimes 1 + u \otimes u + 1 \otimes v,$$

$$w = w \otimes 1 + 3v^2 \otimes u + 6uv \otimes v + 6v \otimes uv + 3u \otimes v^2 + 1 \otimes w.$$

(This corrects the error in 11.6 of [14].)

Proof: We use the method and notation of chapter 2, in particular of the proof of (2.6.5)

The map  $g: V \longrightarrow G_2$  induces  $\mathcal{L}g^*: H^*(\Omega G_2) \longrightarrow \mathcal{L}H^*(V)$ , which identifies  $H^*(\Omega G_2)$  with the rational closure of the subring generated by  $\{1, ig^*PH^*(\Omega G_2)\}$ . The primitive module  $PH^*(\Omega G_2)$  has generators in dimensions 2, 10. By (3.8.2), changing the notation,  $H^*(V)$  has basis

$$\{1, x, \frac{x^2}{3}, \frac{x^3}{6}, \frac{x^4}{18}, \frac{x^5}{18}\} \quad \dim x = 2,$$

so  $H^*(\Omega G_2) \cong$  rational closure of subring generated by  $\{1, i(x), i(x^5)\}$  in  $\mathcal{L}H^*(V)$ .

We have therefore, working always in  $\mathcal{L}^n H^*(V)$  for  $n$  large enough,

$H^0(\Omega G_2)$  is generated by  $1 = 1 \otimes 1 \otimes \dots \otimes 1$ ,

$H^2(\Omega G_2)$  is generated by  $\eta = i(x) = [x]$ ,

In  $H^4(\Omega G_2)$ , we have  $\eta^2 = [x^2] + 2[x \otimes x]$ , and  $3|x^2$  so  $\eta^2$  is not divisible, therefore  $H^4(\Omega G_2)$  is generated by  $\eta^2$ .

In  $H^6(\Omega G_2)$ , we have  $\eta^3 = [x^3] + 3[x^2 \otimes x] + 6[x \otimes x \otimes x]$ , and  $6|x^3$ ,  $3|x^2$  thus  $\eta^3$  is divisible by  $\text{hcf}(6,9,6) = 3$ , thus  $H^6(\Omega G_2)$  is generated by  $\frac{\eta^3}{3}$ .

Similarly  $H^8(\Omega G_2)$  is generated by  $\frac{\eta^4}{6}$ .

In  $H^{10}(\Omega G_2)$ , we have  $18\zeta = i(x^5) = [x^5]$ ,

$$\begin{aligned} \text{and } \eta^5 = & [x^5] + 5[x^4 \otimes x] + 10[x^3 \otimes x^2] \\ & + 20[x^3 \otimes x \otimes x] + 30[x^2 \otimes x^2 \otimes x] \\ & + 60[x^2 \otimes x \otimes x \otimes x] + 120[x \otimes x \otimes x \otimes x \otimes x]. \end{aligned}$$

Thus  $H^{10}(\Omega G_2)$  has basis  $\{\frac{\eta^5 - 18\zeta}{30}, \zeta\}$ .

The diagonal is given by the fact that  $\eta$  and  $\zeta$  are primitive.

The result is now obtained by duality, taking  $u$  dual to  $\eta$ ,  $v$  to  $\eta^2$ , and  $w$  to  $\zeta$ . The relation is given by

$$\langle u^2, \eta^2 \rangle = \langle u \otimes u, \Delta \eta^2 \rangle = 2.$$

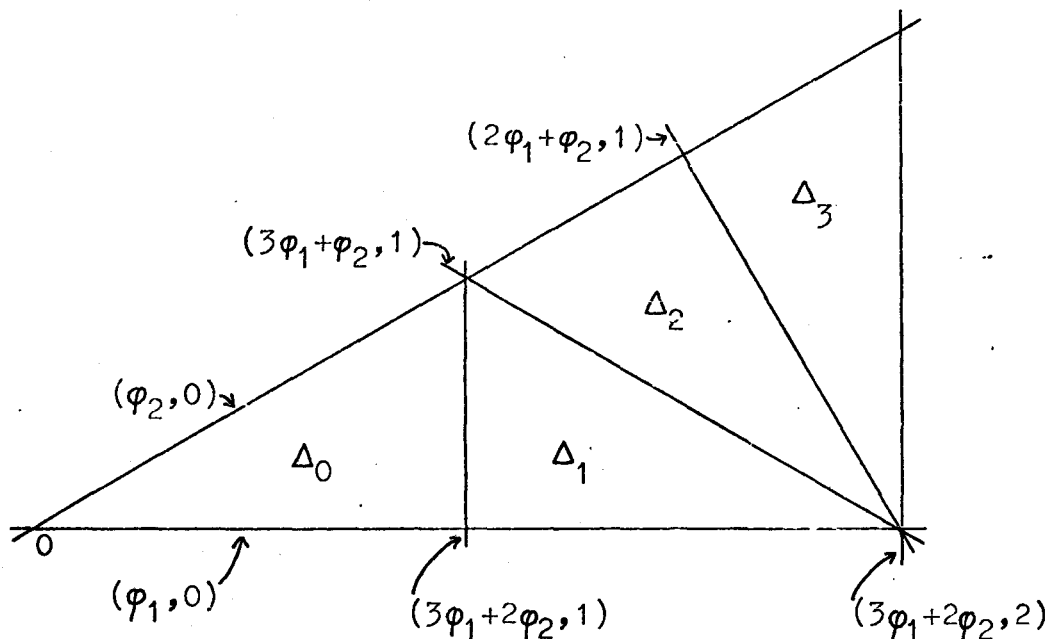
We are now in a position to proceed to the calculation of  $\Omega_*^U(\Omega G_2)$ . We describe now the generators that we shall use.

Recall the following information on the root system of  $G_2$ , see [15] ChIII 14, or [16] for these results.

Positive roots:  $\varphi_2, \varphi_1, \varphi_1+\varphi_2, 2\varphi_1+\varphi_2, 3\varphi_1+\varphi_2, 3\varphi_1+2\varphi_2$   
 with  $3\varphi_1+2\varphi_2$  the dominant root.

Cartan matrix:  $\{n(\varphi_i, \varphi_j)\} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Beginning of the fundamental Weyl chamber  $\mathcal{C}$ :



Here we have used the notation of §3.5 for the oriented singular planes, and numbered the cells of  $\mathcal{C}$   $\Delta_0, \Delta_1, \dots$

We are concerned with the manifold  $\Gamma_{\Delta_2}$  and the map  $f_2: \Gamma_{\Delta_2} \longrightarrow \Omega G_2$ . This will provide a suitable representative for  $V \in \Omega^U(\Omega G_2)$  such that  $\mu V = v \in H_4(\Omega G_2)$ .

Proposition (3.8.4)

$$H^*(\Gamma_{\Delta_2}) = \frac{\mathbb{Z}[x_1, x_2]}{(x_1^2, x_1^2 + x_2^2)}.$$

$$\begin{aligned} c(\Gamma_{\Delta_2}) &= (1+2x_1)(1+2x_2+x_1) \\ &= 1 + (3x_1+2x_2) + 4x_1x_2. \end{aligned}$$

Thus, since the Euler characteristic of  $\Gamma_2$  is 4, we have  $\langle x_1x_2, o_{\Gamma_{\Delta_2}} \rangle = 1$ .

In the notation of (3.8.3) we have

$$f_2^*(\eta) = x_1 + x_2.$$

Proof: Application of Theorem (3.5.1) using the above results on the root system.

### Theorem (3.8.5)

With respect to suitable generators  $U, V, W$  in dimensions

$$2, 4, 10 \quad \Omega_*^U(\Omega G_2) = \frac{\Omega_*^U[U, V, W]}{(U^2 - 2V + 3Y_1U)}$$

Proof: Using (3.8.3) we apply Theorem (3.4.1). In general the ideal must be of the form  $(U^2 - 2V + nY_1U)$ . Thus

$$S_1(U^2 - 2V + nY_1U)(\eta) = 0.$$

Take  $V = [\Gamma_{\Delta_2}, f_2 - 1]$ , then

$$S_1(U^2)(\eta) = (\Delta S_1)(U \otimes U)(\Delta \eta) = 0,$$

$$\begin{aligned} S_1(V)(\eta) &= \langle s_1(\Gamma_{\Delta_2}) \cdot f_2^*(\eta), o_{\Gamma_{\Delta_2}} \rangle, \\ &= \langle (3x_1 + 2x_2)(x_1 + x_2), o_{\Gamma_{\Delta_2}} \rangle, \\ &= 3, \text{ by (3.8.4).} \end{aligned}$$

$$\text{And } S_1(Y_1U)(\eta) = S_1(Y_1) \cdot S_0(U)(\eta) = 2.$$

So we obtain the equation  $-6 + 2n = 0$ , which proves the theorem.



Corollary (3.8.6)

$$K_*(\Omega G_2) = \frac{\mathbb{Z}[\tilde{u}, \tilde{v}, \tilde{w}]}{(\tilde{u}^2 - 2\tilde{v} - 3\tilde{u})}$$

Proof: Conner-Floyd isomorphism.

We proceed now to find the diagonal in  $\mathcal{L}_*^U(\Omega G_2)$ . The proof of (3.8.3) shows that we may take  $W = [G_2/U(2), g-1]$ , where  $g$  is the generating map.

Lemma (3.8.7)

The total Chern class of  $G_2/U(2)$  is

$$c(G_2/U(2)) = 1 + 3x + 13\frac{x^2}{3} + 22\frac{x^3}{6} + 30\frac{x^4}{18} + 6\frac{x^5}{18}.$$

Proof: Consider the diagram

$$\begin{array}{ccccc} G_2/U(2) & \xrightarrow{i} & B_{U(2)} & \longrightarrow & B_{G_2} \\ \uparrow & & \uparrow \rho & & \parallel \\ G_2/T & \longrightarrow & B_T & \longrightarrow & B_{G_2}, \end{array}$$

where  $T \subset U(2) \subset G_2$  is a maximal torus. Let  $\eta$  be the bundle along the fibres of the top row, so that  $i^*\eta$  is the tangent bundle of  $G_2/U(2)$ . Then, by [12] §10.8,

$$\rho^*c(\eta) = \prod_{j \in J} (1 + \epsilon_j b_j),$$

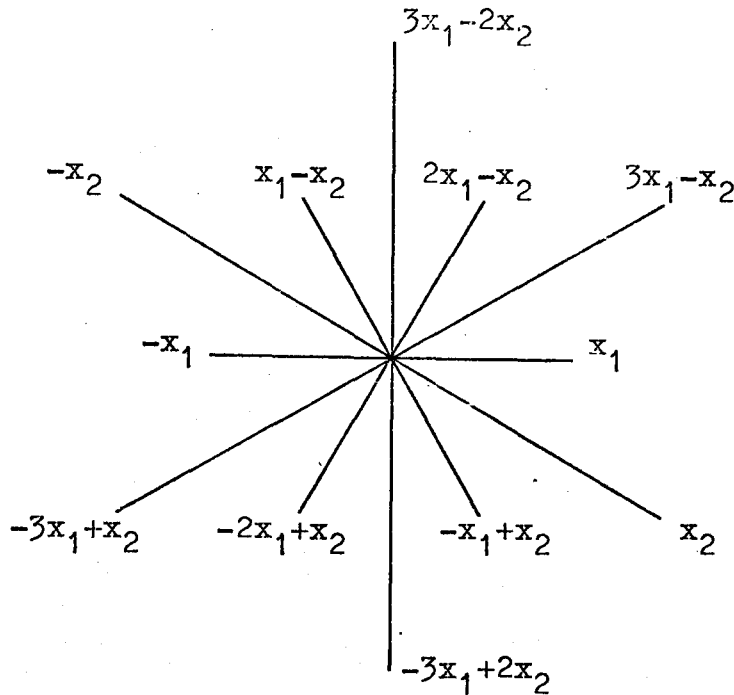
where  $\{\epsilon_j b_j\}_{j \in J}$  are the roots of the complex structure on  $G_2/U(2)$ .

Now  $\rho^*: H^*(B_{U(2)}) \longrightarrow H^*(B_T)$  is the inclusion

$$\mathbb{Z}[y_1, y_2] \subset \mathbb{Z}[x_1, x_2] \text{ where } y_1 = x_2, y_2 = x_1^2 - x_1 x_2,$$

see (3.8.1) and note that the Borel description holds over the integers for  $B_{U(2)}$  since  $U(2)$  is torsion-free, [9].

The star of  $G_2$  is



and the roots of  $U(2) \subset G_2$  are those fixed by  $W(U(2))$  that is  $\{2x_1-x_2, -2x_1+x_2\}$ . The set  $\Psi$  of roots of a complex structure on  $G_2/U(2)$  is characterised by

1)  $\Psi$  is closed, that is if  $\varphi, \theta \in \Psi$  and  $\varphi + \theta$  is a root then  $\varphi + \theta \in \Psi$ .

2)  $\Psi \cup \{2x_1-x_2\}$  is a set of positive roots of  $G_2$  with respect to some ordering on the roots. See [12] §13.7.

Thus, by looking at the star, it is clear that we may take  $\Psi = \{\epsilon_j b_j\}_{j \in J} = \{3x_1-x_2, x_1, x_2, -x_1+x_2, -3x_1-2x_2\}$ .

Therefore  $\rho^*c(\eta) = (1+3x_1-x_2)(1-3x_1+2x_2)(1+x_2)(1+x_1)(1-x_1+x_2)$ .

so that  $c(\eta) = 1 + 3y_1 + y_1^2 - 10y_2 - 3y_1^3 - 20y_1y_2 - 2y_1^4 - 8y_1^2y_2$   
 $+ 9y_2^2 + 2y_1^3y_2 + 9y_1y_2^2. \quad (3.8.8)$

Now  $i^*: H^*(B_{U(2)}) \longrightarrow H^*(G_2/U(2))$  is determined by the image of  $y_1, y_2$ ; suppose  $i^*(y_1) = nx$ ,  $i^*(y_2) = m\frac{x^2}{3}$  where  $m$  and  $n$  are integers. Since  $c(G_2/U(2)) = i^*c(\eta)$ , we have  $c_5(G_2/U(2)) = i^*(2y_1^3y_2 + 9y_1y_2^2) = nm(2n^2 + 3m)\frac{x^5}{3}$ .

But, since the top Chern class evaluated on the orientation class is equal to the Euler characteristic  $\chi(G_2/U(2)) = 6$ , we must have  $c_5(G_2/U(2)) = 6\frac{x^5}{18}$ , changing  $x$  to  $-x$  if necessary. Thus  $nm(2n^2 + 3m) = 1$ , but it is easy to verify that the only integer solution to this equation is  $n = 1$ ,  $m = -1$ . This enables us to compute the result from (3.8.8).

### Theorem (3.8.9)

With respect to the generator  $W = [G_2/U(2), g-1]$  described above, the diagonal in  $\Omega_*^U(\Omega G_2)$  has the form

$$\Delta U = U \otimes 1 + 1 \otimes U,$$

$$\Delta V = V \otimes 1 + U \otimes U + 1 \otimes V,$$

$$\Delta W = W \otimes 1 + 3V^2 \otimes U + 6UV \otimes V + 6V \otimes UV + 3U \otimes V^2 + 1 \otimes W$$

$$- 9Y_1(UV \otimes U + V \otimes V + U \otimes UV)$$

$$+ (18Y_1^2 + 2Y_2)(U \otimes V + U \otimes V)$$

$$- (30Y_1^3 + 9Y_2)(U \otimes U).$$

Proof: Set  $W$  equal to the most general expression,

using §3.4 and Theorem (3.8.3), involving eight unknown coefficients. Then use §3.2 and Lemma (3.8.6) to compute the characteristic numbers and obtain eight equations which determine the unknowns. An additional check is given by requiring  $\Delta$  to be associative, in particular that  $(\Delta \otimes 1)\Delta W = (1 \otimes \Delta)\Delta W$ ; this provides one extra equation.

Corollary (3.8.10)

$K_*(\Omega G_2)$  has diagonal given by

$$\Delta \tilde{u} = \tilde{u} \otimes 1 + 1 \otimes \tilde{u},$$

$$\Delta \tilde{v} = \tilde{v} \otimes 1 + \tilde{u} \otimes \tilde{u} + 1 \otimes \tilde{v},$$

$$\begin{aligned} \Delta \tilde{w} = & \tilde{w} \otimes 1 + 3\tilde{v}^2 \otimes \tilde{u} + 6\tilde{u}\tilde{v} \otimes \tilde{v} + 6\tilde{v} \otimes \tilde{u}\tilde{v} + 3\tilde{u} \otimes \tilde{v}^2 + 1 \otimes \tilde{w} \\ & + 9\tilde{u}\tilde{v} \otimes \tilde{u} + 9\tilde{v} \otimes \tilde{v} + 9\tilde{u} \otimes \tilde{u}\tilde{v} \\ & + 20\tilde{v} \otimes \tilde{u} + 30\tilde{u} \otimes \tilde{u} + 20\tilde{u} \otimes \tilde{v}. \end{aligned}$$

Proof: Conner-Floyd isomorphism.

The relative complexity of the three Hopf algebras is expressed in

Proposition (3.8.11) (see (3.6.8) for notation)

- 1)  $\text{Coker}\{\text{PH}_*(\Omega G_2) \longrightarrow \text{QH}_*(\Omega G_2)\} \cong \mathbb{Z}_2 + \mathbb{Z}_5$  (in dimensions 4 and 10).
- 2) As a graded module over  $\Omega_*^U$ ,  $\text{Coker}\{\text{P}\Omega_*^U(\Omega G_2) \longrightarrow \text{Q}\Omega_*^U(\Omega G_2)\}$  is generated by elements  $a, b$  in dimension 4, 10, subject

to the relations  $2a = 0$ ,  $30b = Y_1^3 a$ .

$$3) \operatorname{Coker}\{PK_*(\Omega G_2) \longrightarrow QK_*(\Omega G_2)\} \cong \mathbb{Z}_{60}.$$

Proof: Follows from (3.8.9), just like (3.6.8).

We end this section by showing how we may deduce the action of the Steenrod operations on  $H^*(\Omega G_2; \mathbb{Z}_p)$  from our results.

Lemma (3.8.12)

Under the Chern character

$$\operatorname{ch}: K_*(\Omega G_2) \longrightarrow H_*(\Omega G_2; \mathbb{Q}) = \mathbb{Q}[u, w], \text{ we have}$$

$$\operatorname{ch} \tilde{u} = u,$$

$$\operatorname{ch} \tilde{v} = \frac{u^2}{2} - \frac{3u}{2},$$

$$\operatorname{ch} \tilde{w} = w - \frac{9u^4}{8} + \frac{10u^3}{3} - \frac{39u^2}{8} + \frac{211u}{60}.$$

Proof: Use (3.1.6) and the known representatives of the bordism elements  $U, V, W$ .

Corollary (3.8.13)

The action of the Adams operations on  $K_0(\Omega G_2)$  is given by  $\psi^k \tilde{u} = k \tilde{u}$ ,

$$\psi^k \tilde{v} = k^2 \tilde{v} + \frac{3k(k-1)}{2} \tilde{u},$$

$$\psi^k \tilde{w} = k^5 \tilde{w} + \frac{9}{2} k^4 (k-1) \tilde{v}^2 + \frac{k^3 (k-1) (41k-40)}{6} \tilde{u} \tilde{v}$$

$$+ \frac{k^2 (k-1) (40k^2 - 41k + 39)}{4} \tilde{v}$$

$$+ \left( \frac{k(k-1)(k-2)(k+1)(1378k-889)}{120} + \frac{80k(k-1)(k-2)}{3} + 35k(k-1) \right) \tilde{u}.$$

Proof: Use the dual of Theorem 5.1 (vi) of [1] to

compute the  $\psi^k$  from (3.8.12).

Now  $K_*(\Omega G_2)_{10}$  has basis  $\{1, \tilde{u}, \tilde{v}, \tilde{u}\tilde{v}, \tilde{v}^2, \tilde{u}\tilde{v}^2, \tilde{w}\}$ , if we take  $\{1, \eta, \zeta\}$  in  $K^*(\Omega G_2)/K^*(\Omega G_2)_{11}$  dual to  $\{1, \tilde{u}, \tilde{w}\}$  in this basis, we find from (3.8.10) that

$$\{1, \eta, \eta^2, \frac{\eta^3 + \zeta}{3}, \frac{\eta^4 + 3\zeta}{6}, \frac{\eta^5 - 18\zeta}{30}, \zeta\}$$

is a basis for  $K^*(\Omega G_2)$  modulo  $K^*(\Omega G_2)_{11}$ .

Proposition (3.8.14)

$$\psi^3(\frac{\eta^3 + \zeta}{3}) = 27(\frac{\eta^3 + \zeta}{3}) - 81(\frac{\eta^5 - 18\zeta}{30}) - 171\zeta, \text{ modulo } K^*(\Omega G_2)_{11}.$$

Proof: By duality from (3.8.13).

Proposition (3.8.15) (Bott-Samelson, [15] ChIII 14.1)

Up to dimension 10,  $H^*(\Omega G_2; \mathbb{Z}_3)$  has basis

$$\{1, \bar{\eta}, \bar{\eta}^2, \bar{\theta}, \bar{\eta}\bar{\theta}, \bar{\eta}^2\bar{\theta}, \bar{\xi}\} \dim \bar{\eta} = 2, \dim \bar{\theta} = 6, \dim \bar{\xi} = 10,$$

with  $\bar{\eta}^3 = 0$ , and  $\bar{\theta}^1 \bar{\theta} = -\bar{\xi}$ .

Proof: The multiplicative structure is immediate since  $H^*(\Omega G_2)$  is torsion-free and so (3.8.12) identifies the skeleton filtration on  $K^*(\Omega G_2)$ . We put

$$\bar{\eta} = \eta \text{ modulo } K^*(\Omega G_2)_3 \text{ reduced mod } 3,$$

$$\bar{\theta} = (\frac{\eta^3 + \zeta}{3}) \text{ modulo } K^*(\Omega G_2)_7 \text{ reduced mod } 3,$$

$$\text{and } \bar{\xi} = \zeta \text{ modulo } K^*(\Omega G_2)_{11} \text{ reduced mod } 3.$$

Then by [7] theorem (6.5) and (3.8.14)

$$\bar{\theta}^1 \bar{\theta} = 9\bar{\eta}^2 \bar{\theta} - 19\bar{\xi} = -\bar{\xi}.$$

Proposition (3.8.16)

Up to dimension 10,  $H^*(\Omega G_2; \mathbb{Z}_2)$  has basis

$\{1, \bar{\eta}, \bar{\eta}^2, \bar{\eta}^3, \bar{\kappa}, \bar{\eta}\bar{\kappa}, \bar{\xi}\}$   $\dim \bar{\eta} = 2$ ,  $\dim \bar{\kappa} = 8$ ,  $\dim \bar{\xi} = 10$ ,  
with  $\bar{\eta}^4 = 0$ , and  $Sq^2 \bar{\kappa} = \bar{\xi}$ .

Proof: As (3.8.15).

This result does not appear to have been known before. It can be used, just as Bott and Samelson use their's, for killing homotopy groups and thus to confirm the known results on the 2-primary component of the homotopy groups of  $G_2$  up to dimension 10.

## References

1. J.F. Adams: Vector fields on spheres, Ann. of Math. (2) 75 (1962), 603-632.
2. J.F. Adams: Lectures on Lie groups, W.A. Benjamin, New York (1969).
3. S. Araki: Hopf structures attached to K-theory; Hodgkin's theorem, Ann. of Math. 85 (1967) 508-525.
4. M.F. Atiyah: Characters and cohomology of finite groups, Inst. Hautes Études Sci. Publ. Math. N° 9 (1961), 23-64.
5. M.F. Atiyah: Vector bundles and the Künneth formula, Topology 1 (1962) 245-248.
6. M.F. Atiyah: On the K-theory of compact Lie groups, Topology 4 (1965), 95-99.
7. M.F. Atiyah: Power operations in K-theory, Quart. J. Math. Oxford Ser. (2) 17 (1966), 165-193.
8. M.F. Atiyah and F. Hirzebruch: Vector bundles and homogeneous spaces, Proc. Sympos. Pure Math. Vol III, 7-38. Amer. Math. Soc. (1961).
9. A. Borel: Sur la cohomologie des espace fibrés principaux et des espaces homogènes de groupes



- de Lie compacts, Ann. of Math. (2) 57 (1953) 115-207.
10. A. Borel: Topology of Lie groups and characteristic classes, Bull. Amer. Math. Soc. 61 (1955), 397-432.
  11. A. Borel et J. De Siebenthal: Les sous-groupes fermés de rang maximum des groupes de Lie clos, Comment. Math. Helv. 23 (1949), 200-221.
  12. A. Borel and F. Hirzebruch: Characteristic classes and homogeneous spaces I, Amer. J. Math. 80 (1958), 458-538.
  13. R. Bott: An application of the Morse theory to the topology of Lie-groups, Bull. Soc. Math. France 84 (1956), 251-281.
  14. R. Bott: The space of loops on a Lie group, Michigan Math. J. 5 (1958), 35-61.
  15. R. Bott and H. Samelson: Applications of the theory of Morse to symmetric spaces, Amer. J. Math. 80 (1958), 964-1029.
  16. N. Bourbaki: Éléments de Mathématiques XXXIV, Groupes et algèbres de Lie, chapitres 4,5 et 6, Hermann, Paris (1968).
  17. V.M. Buhštaber and A.S. Miščenko: Elements of infinite filtration in K-theory, Dokl. Akad.

- Nauk SSSR 178 (1968), 1234-1237. = Soviet Math. Dokl. 9 (1968) 256-259.
18. V.M. Buhštaber and A.S. Miščenko: K-theory on the category of infinite cell complexes, Izv. Akad. Nauk SSSR Ser. Math. 32 (1968), 560-604. = Math. USSR-Izv. 2 (1968), 515-556.
  19. H. Cartan: Algèbres d'Eilenberg-MacLane et homotopie, Séminaire H. Cartan de l'École Normale Supérieure, (1954/1955).
  20. H. Cartan and S. Eilenberg: Homological Algebra, University Press, Princeton (1956).
  21. P.E. Conner and E.E. Floyd: Differentiable periodic maps, Springer-Verlag, Berlin (1964).
  22. P.E. Conner and E.E. Floyd: The relation of cobordism to K-theories, Lecture notes in mathematics 28, Springer-Verlag, Berlin (1966).
  23. P.E. Conner and E.E. Floyd: Torsion in SU bordism, Mem. Amer. Math. Soc. No. 60 (1966).
  24. P.E. Conner and L. Smith: On the complex bordism of finite complexes, Inst. Hautes Etudes Sci. Publ. Math. N° 37 (1969), 117-222.
  25. E. Dyer: Cohomology theories, W.A. Benjamin,

- New York (1969).
26. L. Flatto and Sister M.M. Wiener: Invariants of finite reflection groups and mean value problems, Amer. J. Math. 91 (1969), 591-597.
  27. B. Harris: Torsion in Lie groups and related spaces, Topology 5 (1966), 347-354.
  28. A. Hattori: Integral characteristic numbers for weakly almost complex manifolds, Topology 5 (1966), 259-280.
  29. F. Hirzebruch: Topological methods in algebraic geometry, 3rd edition, Springer-Verlag, Berlin (1966).
  30. W.V.D. Hodge and D. Pedoe: Methods of algebraic geometry, Cambridge University Press (1947-1954).
  31. L. Hodgkin: On the K-theory of Lie groups, Topology 6 (1967), 1-36.
  32. L. Hodgkin: An equivariant Künneth formula in K-theory, Warwick University preprint (1968).
  33. L. Hodgkin: Notes towards a geometric Eilenberg-Moore sequence, preprint, Forschungsinstitut für Mathematik, ETH, Zürich (1968).
  34. I.M. James: Ex-homotopy theory, Illinois J. Math.

- 15 (1971), 324-337.
35. P.S. Landweber: Künneth formulas for bordism theories, Trans. Amer. Math. Soc. 121 (1966), 242-256.
  36. S. MacLane: Homology, Springer-Verlag, Berlin (1963).
  37. J. Milnor: Lectures on characteristic classes, Duplicated notes, Princeton (1957).
  38. J. Milnor: On axiomatic homology theory, Pacific J. Math. 12 (1962), 337-341.
  39. J. Milnor and J.C. Moore: On the structure of Hopf algebras, Ann. of Math. (2) 81 (1965) 211-264.
  40. T. Petrie: The weakly complex bordism of Lie groups, Ann. of Math. 88 (1968), 371-402.
  41. J.-E. Roos: Sur les foncteurs dérivés de  $\varprojlim$ . Applications, C.R. Acad. Sci. Paris 252 (1961), 3702-3704.
  42. G. Segal: Classifying spaces and spectral sequences, Inst. Hautes Études Sci. Publ. Math. N° 34 (1968), 105-112.
  43. J.-P. Serre: Homologie singulière des espaces fibrés. Applications, Ann. of Math. 54 (1951) 425-505.
  44. J.-P. Serre: Réprésentations linéaires et espaces homogènes kähleriens des groupes de Lie compacts,

- Seminaire Bourbaki N° 100, Paris (1954).
45. L. Smith: Homological algebra and the Eilenberg-Moore spectral sequence, Trans. Amer. Math. Soc. 129 (1967), 58-93.
  46. L. Smith: Lectures on the Eilenberg-Moore spectral sequence, Lecture notes in mathematics 134, Springer-Verlag, Berlin (1970).
  47. L. Smith: On realising complex bordism modules II, Applications to the stable homotopy groups of spheres, Amer. J. Math. 93 (1971) 226-263.
  48. V.P. Snaith: On the K-theory of homogeneous spaces and conjugate bundles, Proc. London Math. Soc. (3) 22 (1971), 562-584.
  49. E.H. Spanier: Algebraic Topology, McGraw Hill, New York (1966).
  50. R.E. Stong: Notes on cobordism theories, University Press, Princeton (1968).
  51. R. Thom: Travaux de Milnor sur le cobordisme, Seminaire Bourbaki N° 180, Paris (1959).
  52. G.W. Whitehead: Generalised homology theories, Trans. Amer. Math. Soc. 102 (1962), 227-283.